LEZIONE 10

Teoremi generali sui limiti

1° Conseguenze logiche della definizione di limite

$$\lim_{x \to c} f(x) = l \implies \lim_{x \to c} [-f(x)] = -l$$

$$\boxed{|f(x) - l| < \varepsilon} \qquad |[-f(x)] - [-l]| < \varepsilon$$

2° $|f(x) - l| < \varepsilon$

Aggiungo e sottraggo una quantità costante A

$$|(f(x) - A) + A - l| < \varepsilon \implies |\underline{(f(x) - A)} - \underline{(l - A)}| < \varepsilon$$

$$\lim_{\substack{x \to c \\ [\pm\infty]}} [f(x) - A] = l - A$$

$$\boxed{A = l}$$

Se la costante è uguale al valore limite

$$\lim_{\substack{x \to c \\ [\pm\infty]}} f(x) - l = l - l = 0 \implies \boxed{\lim_{\substack{x \to c \\ [\pm\infty]}} [f(x) - l] = 0}$$

TEOREMA DI UNICITA' DI LIMITE

IP $\exists \lim_{x \to c} f(x) = l$ $l \in \mathbb{R}$ [SI PUÒ ESTENDERE CON $l \to \pm\infty$]

TS $\exists! \lim_{x \to c} f(x) = l$

DIM

SUPP. PER ASSURDO $\exists\ l, l' : l \neq l'$

ε INFINT. $\Longrightarrow \frac{\varepsilon}{2}$ INFINT.

$\exists\, \ell, \ell'$

$\Downarrow$

$$\forall \frac{\varepsilon}{2} > 0\ \exists I(c)\ \forall x \in I(c) \Longrightarrow |f(x)-\ell| < \frac{\varepsilon}{2}$$

$$\Longrightarrow |f(x)-\ell'| < \frac{\varepsilon}{2}$$

$$\begin{cases} |f(x)-\ell| < \frac{\varepsilon}{2} \\ |f(x)-\ell'| < \frac{\varepsilon}{2} \end{cases}$$

$$\overline{|f(x)-\ell| + |f(x)-\ell'| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon}$$

$$\boxed{|f(x)-\ell|+|f(x)-\ell'|<\varepsilon}$$

$$\boxed{|\ell'-\ell|<\varepsilon}$$

LIMITI DISTINTI MA VICINI

UTILIZZIAMO QUESTA IDENTITÀ

$$\boxed{[f(x)-\ell]-[f(x)-\ell']=\ell'-\ell}$$

$$\Downarrow$$

$$\ell'-\ell \leq |f(x)-\ell|+|f(x)-\ell'|$$

$$\Downarrow$$

$$\boxed{\varepsilon \leq |f(x)-\ell|+|f(x)-\ell'|}$$

ASSURDO!

C.V.D

TEOREMA DELLA PERMANENZA DEL SEGNO

Ip $\boxed{\lim_{x \to c} f(x) = l}$ $\quad l \neq 0$

Ts $\quad \exists\, I(c) \setminus \{c\} \Rightarrow \begin{cases} l > 0 \\ f(x) > 0 \end{cases} \wedge \begin{cases} l < 0 \\ f(x) < 0 \end{cases}$

$c \Rightarrow \pm\infty$ $\quad l \to \pm\infty$

$[\infty]$ $\quad l \neq \infty$

DIM PER IPOTESI

$$\lim_{x \to c} f(x) = l \qquad l \neq 0$$

$$\forall \varepsilon > 0 \ \exists I(c) : \forall x \in I(c) \setminus \{c\} \Longrightarrow |f(x) - l| < \varepsilon$$

$$-\varepsilon < f(x) - l < \varepsilon$$

SOMMANDO l AI 3 MEMBRI

$$\boxed{l - \varepsilon < f(x) < l + \varepsilon}$$

SCELGO $\boxed{\varepsilon = \frac{|l|}{2}}$

$$\underbrace{l - \frac{|l|}{2} < f(x)}_{\circ} \underbrace{< l + \frac{|l|}{2}}_{\circ} \Longrightarrow \begin{cases} f(x) > l - \frac{|l|}{2} \\ f(x) < l + \frac{|l|}{2} \end{cases}$$

se $\underline{l > 0}$ $\quad |l| = l$

$$\begin{cases} f(x) > l - \frac{l}{2} = \frac{l}{2} \\ f(x) < l + \frac{l}{2} = \frac{3l}{2} \Rightarrow f(x) < \frac{3l}{2} \end{cases}$$

$f(x) > \frac{l}{2} \Longrightarrow f(x) > 0$
$\forall x \in I(c) \setminus \{c\}$

$\underline{l < 0}$ $\quad |l| = -l$

$$\begin{cases} f(x) > l + \dfrac{l}{2} \Rightarrow f(x) > \dfrac{3l}{2} \quad ?!? \\ f(x) < l - \dfrac{l}{2} = \dfrac{l}{2} \quad \checkmark \end{cases}$$

$$f(x) < \frac{l}{2} \quad \forall x \in I(c) \setminus \{c\}$$

$\underline{\text{c.v.d}}$

$$\boxed{\begin{array}{l} \text{Ip } \forall x \in I(c)\setminus\{c\} \Rightarrow f(x) \geq 0 \\ \text{Ts } \quad l \geq 0 \qquad \lim_{x\to c} f(x) = l \end{array}}$$

Dim

PER ASSURDO $\Rightarrow \lim_{x\to c} f(x) = l < 0$

PER IL TEOREMA DELLA PERMANENZA DEL SEGNO

$\exists\, I(c)\setminus\{c\} \Rightarrow f(x) < 0$ ASSURDO

MA PER IPOTESI $\forall x \in I(c)\setminus\{c\} \Rightarrow f(x) \geq 0$

c.v.d

Ovviamente si possono tranquillamente invertire l'ipotesi e la tesi del teorema precedente

1° teorema del confronto

Ip $\lim_{x \to c} f(x) = l$; $\lim_{x \to c} g(x) = l$

$\varphi(x) : \forall x \in I(c) \setminus \{c\} \Longrightarrow f(x) \leq \varphi(x) \leq g(x)$

Ts $\lim_{x \to c} \varphi(x) = l$

DIM $\boxed{\forall x \in I_1(c)\setminus\{c\} \Longrightarrow f(x) \le e(x) \le g(x)}$ (A)

PER IPOTESI

PER IPOTESI $\forall \varepsilon > 0$

$\lim_{x \to c} f(x) = \ell$ $\quad \exists I_2(c)\setminus\{c\} \Longrightarrow |f(x) - \ell| < \varepsilon$

$\boxed{\ell - \varepsilon < f(x) < \ell + \varepsilon}$ (B)

$\lim_{x \to c} g(x) = \ell$ $\quad \forall \varepsilon > 0 \ \exists I_3(c)\setminus\{c\} \Longrightarrow |g(x) - \ell| < \varepsilon$

$\boxed{\ell - \varepsilon < g(x) < \cancel{\ell + \varepsilon}}$ (C)

$$\boxed{I = I_1 \cap I_2 \cap I_3}$$

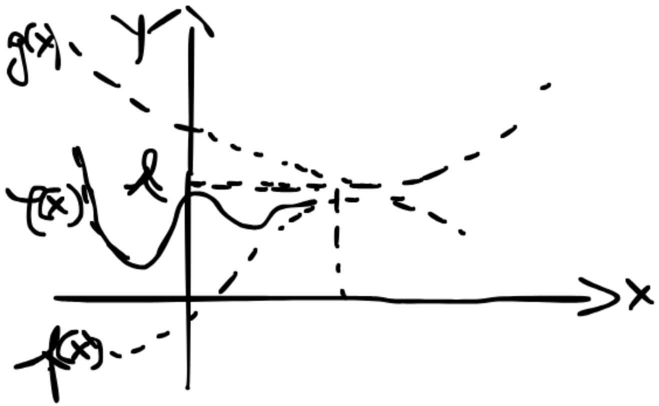
$$\forall x \in I \setminus \{c\} \Rightarrow l-\varepsilon < f(x) \le \varphi(x) \le g(x) < l-\varepsilon$$

$$\Downarrow$$

$$\boxed{l-\varepsilon < \varphi(x) < l+\varepsilon}$$

$$|\varphi(x) - l| < \varepsilon \Longrightarrow \exists \lim_{x \to c} \varphi(x) = l$$

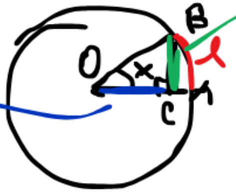
c.v.d.


g(x)
y
f(x)
ℓ
x
h(x)

$$\boxed{\lim_{x \to 0^+} \operatorname{sen} x = 0}$$

Vogliamo dimostrarlo!!!

$\cos x \; (r=1)$



$\operatorname{sen} x \; (r=1)$

$\overline{BC} < \overset{\frown}{AB}$

$\overline{OA} = \overline{OB} = r$

$A\hat{O}B \quad x$ angolo in radianti

$\overset{\frown}{AB}$ arco relativo ad $A\hat{O}B$

$\overset{\frown}{AB} = l$

$$I^+(x=0) = \left]0, \frac{\pi}{2}\right[\Rightarrow 0 < x < \frac{\pi}{2}$$

DALLA DEF. DI RADIANTE $\boxed{x = \frac{l}{r}} \Rightarrow \frac{\overset{\frown}{AB}}{r} = x$

$$\frac{\overline{BC}}{r} < \frac{\overline{AB}}{r}$$

$$\boxed{0 \leq x < \frac{\pi}{2}}$$

$$\boxed{0 < \operatorname{sen} x < x}$$

DIVIDO PER R

$$\frac{\overline{BC}}{r} = \frac{\overline{BC}}{\overline{OB}} = \operatorname{sen} x$$

$$\frac{\overline{AB}}{r} = \frac{l}{r} = x$$

DAL 1° TEOREMA DEI CONFRONTI

APPLICANDO IL TEOR. DEL CONFRONTO

$f(x) = \operatorname{sen} x$

$h(x) = 0$

$$\lim_{x \to 0} 0 = 0$$

$g(x) = x$

$$\lim_{x \to 0} x = 0 = f(0)$$

$$\boxed{\lim_{x \to 0^+} \operatorname{sen} x = 0}$$

2º TEOREMA DEL CONFRONTO

Ip $\forall x \in I(c) \setminus \{c\}$

$|f(x)| \leq g(x)$: $\lim_{x \to c} g(x) = 0$

Ts $\lim_{x \to c} f(x) = 0$

DIM PER IPOTESI

$\forall x \in I_1(c) \setminus \{c\} \Rightarrow |g(x) - 0| < \varepsilon \Rightarrow \underline{|g(x)| < \varepsilon}$

$\boxed{\lim_{x \to c} g(x) = 0}$

$\forall x \in I_2(c) \setminus \{c\} \Rightarrow \underline{|f(x)| \leq |g(x)|}$

$I = I_1 \cap I_2$

$\forall x \in I = I_1 \cap I_2 \setminus \{c\} \Rightarrow |f(x)| \leq |g(x)| < \varepsilon \Rightarrow |f(x)| < \varepsilon$

$\lim_{x \to c} f(x) = 0$

$$\lim_{x \to 0} \left(x \operatorname{sen} \frac{1}{x} \right) = 0$$

SAPPIAMO $\quad -1 \le \operatorname{sen} x \le 1 \Rightarrow -1 \le \operatorname{sen} \frac{1}{x} \le 1$

$$\Downarrow$$

$$\Rightarrow \boxed{\left| \operatorname{sen} \frac{1}{x} \right| \le 1} \quad \forall x \ne 0$$

MOLTIPLICO PER $|x|$ LA $\Rightarrow$

$$|x| \cdot \left| \operatorname{sen} \frac{1}{x} \right| \le |x| \Rightarrow \underbrace{\left| x \cdot \operatorname{sen} \frac{1}{x} \right|}_{\lim_{x \to 0} x \operatorname{sen} \frac{1}{x} = 0} \le \underbrace{|x|}_{\lim_{x \to 0} x = 0}$$

DAL TEOR DEL CONFRONTO

3° teorema del confronto

Ip $\forall x \in I(c) \setminus \{c\} \Longrightarrow |g(x)| \geq |f(x)|$

$$\lim_{x \to c} f(x) = \infty$$

Ts $$\lim_{x \to c} g(x) = \infty$$

DIM

$\forall x \in I_1(c) \Longrightarrow |f(x)| > M$

$\lim_{x \to c} f(x) = \infty$

$\forall x \in I_2(c) \Longrightarrow |g(x)| \geq |f(x)|$

M
VALORE
INF.
GRANDE

$I = I_1(c) \cap I_2(c)$

$\forall x \in I(c) \setminus \{c\} \Longrightarrow |g(x)| \geq |f(x)| > M \Longrightarrow |g(x)| > M$

$\forall x \in I(c) \setminus \{c\} \Longrightarrow \boxed{\lim_{x \to c} g(x) = \infty}$

$$\boxed{\lim_{x \to +\infty} \left[2^x (\operatorname{sen} x + 2)\right]} = ?$$

$\operatorname{sen}(x) \geq -1$ AGGIUNGO 2 AD AMBO I MEMBRI

$\operatorname{sen}(x) + 2 \geq -1 + 2 \implies \boxed{\operatorname{sen}(x) + 2 \geq 1}$ ⊗

MOLTIPLICO A PER 2^x

$$\boxed{2^x \cdot (\operatorname{sen} x + 2) \geq 2^x}$$

APPLICO I ~~VALORI~~ ASSOLUTI A ENTRAMBI I MEMBRI

$$|2^x (\operatorname{sen} x + 2)| \geq |2^x|$$

$$|g(x)| \geq |f(x)|$$

$$\lim_{x \to +\infty} 2^x = +\infty \implies \lim_{x \to +\infty} 2^x (\operatorname{sen} x + 2) = +\infty$$

DAL 2^a TEOREMA DEL CONFRONTO.