LEZIONE 11

Espressioni goniometriche contenenti formule di duplicazione e bisezione

DUPLICAZ.

$\operatorname{sen} 2\alpha = 2 \operatorname{sen}\alpha \cos\alpha$

$\cos 2\alpha = \cos^2\alpha - \operatorname{sen}^2\alpha$

$\boxed{\cos^2\alpha = 1 - \operatorname{sen}^2\alpha}$

$$\operatorname{sen}^2 2\alpha + \cos 2\alpha + 4\operatorname{sen}^2\alpha =$$
$$= (2\operatorname{sen}\alpha\cos\alpha)^2 + \cos^2\alpha - \operatorname{sen}^2\alpha + 4\operatorname{sen}^2\alpha =$$
$$= 4\operatorname{sen}^2\alpha\cos^2\alpha + \cos^2\alpha - \operatorname{sen}^2\alpha + 4\operatorname{sen}^2\alpha =$$
$$= 4\operatorname{sen}^2\alpha\cos^2\alpha + \cos^2\alpha + 3\operatorname{sen}^2\alpha =$$
$$= 4\operatorname{sen}^2\alpha \cdot (1 - \operatorname{sen}^2\alpha) + 1 - \operatorname{sen}^2\alpha + 3\operatorname{sen}^2\alpha =$$
$$= \underline{4\operatorname{sen}^2\alpha} - 4\operatorname{sen}^4\alpha + 1 - \underline{\operatorname{sen}^2\alpha} + \underline{3\operatorname{sen}^2\alpha} = 6\operatorname{sen}^2\alpha - 4\operatorname{sen}^4\alpha + 1$$

$$\boxed{\cos^2\alpha = 1-\operatorname{sen}^2\alpha}$$

$$\operatorname{sen}^2 2\alpha + \cos 2\alpha + 4\operatorname{sen}^4\alpha =$$

$$= (2\operatorname{sen}\alpha\cos\alpha)^2 + (\cos^2\alpha - \operatorname{sen}^2\alpha) + 4\operatorname{sen}^4\alpha =$$

$$= 4\operatorname{sen}^2\alpha\cos^2\alpha + 1 - \operatorname{sen}^2\alpha - \operatorname{sen}^2\alpha + 4\operatorname{sen}^4\alpha$$

$$= 4\operatorname{sen}^2\alpha(1-\operatorname{sen}^2\alpha) + 1 - 2\operatorname{sen}^2\alpha + 4\operatorname{sen}^4\alpha$$

$$= 4\operatorname{sen}^2\alpha - \cancel{4\operatorname{sen}^4\alpha} + 1 - 2\operatorname{sen}^2\alpha + \cancel{4\operatorname{sen}^4\alpha} =$$

$$= \boxed{1 + 2\operatorname{sen}^2\alpha}$$

$$(1+\cos 2\alpha)\cdot \operatorname{tg}\alpha =$$

$$\boxed{\cos 2\alpha = \cos^2\alpha - \operatorname{sen}^2\alpha}$$

$$\boxed{\operatorname{tg}\alpha = \frac{\operatorname{sen}\alpha}{\cos\alpha}}$$

$$\boxed{\operatorname{sen}^2\alpha = 1-\cos^2\alpha}$$

$$=\left[1+\cos^2\alpha - \operatorname{sen}^2\alpha\right]\cdot \frac{\operatorname{sen}\alpha}{\cos\alpha} =$$

$$=\left[1+\cos^2\alpha - (1-\cos^2\alpha)\right]\cdot \frac{\operatorname{sen}\alpha}{\cos\alpha} =$$

$$=\left[\cancel{1}+\cos^2\alpha \cancel{-1}+\cos^2\alpha\right]\cdot \frac{\operatorname{sen}\alpha}{\cos\alpha} = 2\cos^{\cancel{2}}\alpha\cdot \frac{\operatorname{sen}\alpha}{\cancel{\cos\alpha}} = 2\operatorname{sen}\alpha\cos\alpha = \operatorname{sen}2\alpha$$

$$\operatorname{sen} 2\alpha(\operatorname{tg}\alpha + \operatorname{ctg}\alpha)$$

$$(2\operatorname{sen}\alpha\cos\alpha)\cdot\left(\frac{\operatorname{sen}\alpha}{\cos\alpha} + \frac{\cos\alpha}{\operatorname{sen}\alpha}\right) =$$

$$= (2\operatorname{sen}\alpha\cos\alpha)\cdot\left(\frac{\operatorname{sen}^2\alpha + \cos^2\alpha}{\operatorname{sen}\alpha\cos\alpha}\right)$$

$$= 2\,\cancel{\operatorname{sen}\alpha}\cdot\cancel{\cos\alpha}\cdot\frac{1}{\cancel{\operatorname{sen}\alpha}\cdot\cancel{\cos\alpha}} = \boxed{2}$$

$$\boxed{\operatorname{sen} 2\alpha = 2\operatorname{sen}\alpha\cos\alpha}$$

$$\boxed{\begin{array}{l}\operatorname{tg}\alpha = \dfrac{\operatorname{sen}\alpha}{\cos\alpha} \\ \operatorname{ctg}\alpha = \dfrac{\cos\alpha}{\operatorname{sen}\alpha}\end{array}}$$

$$\boxed{\operatorname{sen}^2\alpha + \cos^2\alpha = 1}$$

$$\frac{1-\cos 2\alpha}{2}\left(\operatorname{cotg}^2\alpha-1\right)=$$

$$=\frac{1-\left(\cos^2\alpha-\operatorname{sen}^2\alpha\right)}{2}\cdot\left(\frac{\cos^2\alpha}{\operatorname{sen}^2\alpha}-1\right)=$$

$$=\frac{1-\cos^2\alpha+\operatorname{sen}^2\alpha}{2}\cdot\left(\frac{\cos^2\alpha-\operatorname{sen}^2\alpha}{\operatorname{sen}^2\alpha}\right)=$$

$$=\frac{1-\left(1-\operatorname{sen}^2\alpha\right)+\operatorname{sen}^2\alpha}{2}\cdot\frac{\cos^2\alpha-\operatorname{sen}^2\alpha}{\operatorname{sen}^2\alpha}=\frac{\operatorname{sen}^2\alpha+\operatorname{sen}^2\alpha}{2}\cdot\frac{\cos^2\alpha-\operatorname{sen}^2\alpha}{\operatorname{sen}^2\alpha}$$

$$=\frac{\cancel{2}\,\cancel{\operatorname{sen}^2\alpha}}{\cancel{2}}\cdot\frac{\cos^2\alpha-\operatorname{sen}^2\alpha}{\cancel{\operatorname{sen}^2\alpha}}$$

$$\boxed{\cos 2\alpha}$$

$$\cos 2\alpha=\cos^2\alpha-\operatorname{sen}^2\alpha$$

$$\operatorname{cotg}\alpha=\frac{\cos\alpha}{\operatorname{sen}\alpha}$$

$$\cos^2\alpha=1-\operatorname{sen}^2\alpha$$

$$\boxed{\cos^2\alpha-\operatorname{sen}^2\alpha=\cos 2\alpha}$$

$$2\,tg^2\alpha\,sen^2\frac{\alpha}{2} + tg\alpha\,sen\alpha$$

$$\boxed{sen\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}}}$$

$$\boxed{tg\alpha = \frac{sen\alpha}{\cos\alpha}}$$

$$2\cdot\frac{sen^2\alpha}{\cos^2\alpha}\cdot\left(\pm\sqrt{\frac{1-\cos\alpha}{2}}\right)^2 + \frac{sen\alpha}{\cos\alpha}\cdot sen\alpha =$$

$$= \cancel{2}\,\frac{sen^2\alpha}{\cos^2\alpha}\cdot\left(\frac{1-\cos\alpha}{\cancel{2}}\right) + \frac{sen^2\alpha}{\cos\alpha} =$$

$$= \frac{sen^2\alpha\,(1-\cos\alpha)}{\cos^2\alpha} + \frac{sen^2\alpha}{\cos\alpha} = \frac{sen^2\alpha - \cancel{sen^2\alpha\cos\alpha} + \cancel{sen^2\alpha\cos\alpha}}{\cos^2\alpha}$$

$$= \frac{sen^2\alpha}{\cos^2\alpha} = \left(\frac{sen\alpha}{\cos\alpha}\right)^2 = \boxed{tg^2\alpha}$$

$$\frac{1}{1-tg^2\frac{\alpha}{2}} - \frac{1}{1-ctg^2\frac{\alpha}{2}} =$$

$$tg\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}$$

$$= \frac{1}{1-\left(\frac{1-\cos\alpha}{1+\cos\alpha}\right)} - \frac{1}{1-\left(\frac{1+\cos\alpha}{1-\cos\alpha}\right)} =$$

$$ctg\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{1-\cos\alpha}}$$

$$= \frac{1}{\frac{\cancel{1}+\cos\alpha-\cancel{1}+\cos\alpha}{1+\cos\alpha}} - \frac{1}{\frac{\cancel{1}-\cos\alpha-\cancel{1}-\cos\alpha}{1-\cos\alpha}} =$$

$$= \frac{\cancel{2}}{\cancel{2}\cos\alpha} = \boxed{\frac{1}{\cos\alpha}}$$

$$= \frac{1}{\frac{2\cos\alpha}{1+\cos\alpha}} - \frac{1}{\frac{-2\cos\alpha}{1-\cos\alpha}} = \frac{1+\cos\alpha}{2\cos\alpha} + \frac{1-\cos\alpha}{2\cos\alpha} = \frac{1+\cancel{\cos\alpha}+1-\cancel{\cos\alpha}}{2\cos\alpha}$$

Formule di prostaferesi

Permettono di riscrivere somme di seni o coseni di angoli diversi nel prodotto di seni o coseni.

$$\text{sen}(\alpha+\beta) = \text{sen}\,\alpha\cos\beta + \text{sen}\,\beta\cos\alpha$$

$$\text{sen}(\alpha-\beta) = \text{sen}\,\alpha\cos\beta - \text{sen}\,\beta\cos\alpha$$

$$\boxed{\begin{aligned} \text{sen}(\alpha+\beta) + \text{sen}(\alpha-\beta) &= 2\,\text{sen}\,\alpha\cos\beta \\ \text{sen}(\alpha+\beta) - \text{sen}(\alpha-\beta) &= 2\,\text{sen}\,\beta\cos\alpha \end{aligned}}$$

SOMMO
MEMBRO A MEMBRO

SOTTRAGGO
MEMBRO A MEMBRO

$$\operatorname{sen}(\alpha+\beta)+\operatorname{sen}(\alpha-\beta)=2\operatorname{sen}\alpha\cos\beta$$
$$\operatorname{sen}(\alpha+\beta)-\operatorname{sen}(\alpha-\beta)=2\operatorname{sen}\beta\cos\alpha$$

$$\boxed{\operatorname{sen}p+\operatorname{sen}q=2\operatorname{sen}\left(\frac{p+q}{2}\right)\cos\left(\frac{p-q}{2}\right)}$$
$$\boxed{\operatorname{sen}p-\operatorname{sen}q=2\operatorname{sen}\left(\frac{p-q}{2}\right)\cos\left(\frac{p+q}{2}\right)}$$

$$\begin{cases}\alpha+\beta=p\\ \alpha-\beta=q\end{cases}$$

$$\begin{cases}\alpha=p-\beta\\ p-\beta-\beta=q\end{cases}$$

$$-2\beta=q-p$$

$$\beta=\frac{p-q}{2}$$

$$\alpha=p-\frac{p-q}{2}=\frac{p+q}{2}$$

Formule di Prostaferesi per il seno

$$\cos(\alpha+\beta) = \cos\alpha\cos\beta - \operatorname{sen}\alpha\operatorname{sen}\beta$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \operatorname{sen}\alpha\operatorname{sen}\beta$$

$$\boxed{\begin{aligned}\cos(\alpha+\beta)+\cos(\alpha-\beta) &= 2\cos\alpha\cos\beta\\ \cos(\alpha+\beta)-\cos(\alpha-\beta) &= -2\operatorname{sen}\alpha\cos\beta\end{aligned}}$$

$$\boxed{\begin{aligned}\cos p + \cos q &= 2\cos\left(\frac{p+q}{2}\right)\cos\left(\frac{p-q}{2}\right)\\ \cos p - \cos q &= -2\operatorname{sen}\left(\frac{p+q}{2}\right)\cos\left(\frac{p-q}{2}\right)\end{aligned}}$$

SOMMO

SOTTRAGGO

$\alpha+\beta = p$

$\alpha-\beta = q$

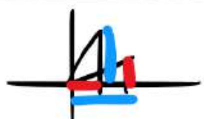
$\alpha = \frac{p+q}{2}$

$\beta = \frac{p-q}{2}$

Formule di Prostaferesi per il coseno

$$\boxed{\operatorname{tg}\alpha \pm \operatorname{tg}\beta = \frac{\operatorname{sen}\alpha}{\cos\alpha} \pm \frac{\operatorname{sen}\beta}{\cos\beta} = \frac{\operatorname{sen}\alpha\cos\beta \pm \operatorname{sen}\beta\cos\alpha}{\cos\alpha\cos\beta} = \frac{\operatorname{sen}(\alpha\pm\beta)}{\cos\alpha\cos\beta}}$$

$$\operatorname{sen}\alpha + \cos\beta = \underbrace{\operatorname{sen}\alpha + \operatorname{sen}(90^\circ - \beta)}_{\text{APPLICO PROSTAFERESI}}$$



$$\boxed{\cos\beta = \operatorname{sen}(90^\circ - \beta)}$$

ESEMPIO

$1 + \operatorname{sen} x = \operatorname{sen} 90° + \operatorname{sen} x =$

$= 2 \cdot \operatorname{sen}\left(\frac{90° + x}{2}\right) \cdot \cos\left(\frac{90° - x}{2}\right) =$

$= 2 \operatorname{sen}\left(45° + \frac{x}{2}\right) \cdot \cos\left(45° - \frac{x}{2}\right) =$

$= 2 \operatorname{sen}\left(45° + \frac{x}{2}\right) \operatorname{sen}\left(45° + \frac{x}{2}\right) =$

$= \boxed{2 \operatorname{sen}^2\left(45° + \frac{x}{2}\right)}$

$p = 90° \quad q = x$

$\operatorname{sen} p + \operatorname{sen} q =$

$= 2 \operatorname{sen}\left(\frac{p+q}{2}\right) \cos\left(\frac{p-q}{2}\right)$

$\cos\left(45° - \frac{x}{2}\right)$

$= \operatorname{sen}\left[90° - \left(45° - \frac{x}{2}\right)\right]$

$A^2 - B^2 = (A+B)(A-B)$

$\operatorname{sen} p + \operatorname{sen} q = 2 \operatorname{sen} \frac{p+q}{2} \cos \frac{p-q}{2}$

$\operatorname{sen} p - \operatorname{sen} q = 2 \operatorname{sen} \frac{p-q}{2} \cos \frac{p+q}{2}$

$$\operatorname{sen}^2 5\alpha - \operatorname{sen}^2 3\alpha =$$

$$= (\operatorname{sen} 5\alpha + \operatorname{sen} 3\alpha)\cdot(\operatorname{sen} 5\alpha - \operatorname{sen} 3\alpha) =$$

$$= \left[2 \operatorname{sen}\left(\frac{5\alpha+3\alpha}{2}\right)\cdot \cos\left(\frac{5\alpha-3\alpha}{2}\right)\right]\cdot\left[2 \operatorname{sen}\left(\frac{5\alpha-3\alpha}{2}\right)\cos\left(\frac{5\alpha+3\alpha}{2}\right)\right]$$

$$= \left[2 \operatorname{sen} \frac{\cancel{8}^{4}\alpha}{\cancel{2}_{1}} \cos \frac{\cancel{2}\alpha}{\cancel{2}}\right]\cdot\left[2 \operatorname{sen} \frac{\cancel{2}\alpha}{\cancel{2}}\cdot \cos \frac{\cancel{8}^{4}\alpha}{\cancel{2}}\right]$$

$$= 4 \operatorname{sen} 4\alpha \cos\alpha \cdot \operatorname{sen}\alpha \cdot \cos 4\alpha =$$

$$= 2 \operatorname{sen}\alpha \cos\alpha \cdot 2 \operatorname{sen} 4\alpha \cos 4\alpha = \boxed{\operatorname{sen} 2\alpha \cdot \operatorname{sen} 8\alpha}$$

Formule di Werner

Sono le formule inverse di prostaferesi

$$\operatorname{sen}\alpha\cos\beta = \frac{1}{2}\left[\operatorname{sen}(\alpha+\beta)+\operatorname{sen}(\alpha-\beta)\right]$$

$$\operatorname{sen}\beta\cos\alpha = \frac{1}{2}\left[\operatorname{sen}(\alpha+\beta)-\operatorname{sen}(\alpha-\beta)\right]$$

$$\cos\alpha\cos\beta = \frac{1}{2}\left[\cos(\alpha+\beta)+\cos(\alpha-\beta)\right]$$

$$\operatorname{sen}\alpha\cos\beta = -\frac{1}{2}\left[\cos(\alpha+\beta)-\cos(\alpha-\beta)\right] = \frac{1}{2}\left[\cos(\alpha-\beta)-\cos(\alpha+\beta)\right]$$