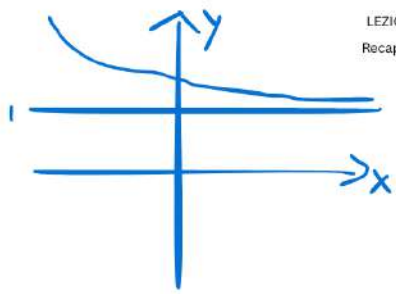
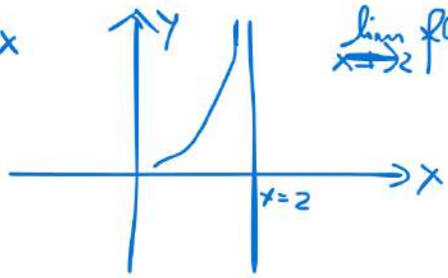
LEZIONE 11

Recap veloce



$$\lim_{x \to +\infty} f(x) = 1$$



$$\lim_{x \to 2} f(x) = +\infty$$

$\varepsilon = 0{,}1$

$0{,}01$

FUNZIONE CONTINUA

Una funzione si dice continua quando si verifica questa condizione:

$$\boxed{\lim_{x \to c} f(x) = f(c)}$$

es $\boxed{\lim_{x \to 2} x^2 = f(2) = 4}$

$$\boxed{\forall \varepsilon > 0 \ \exists I(c) : \forall x \in I(c) \Longrightarrow |f(x) - f(c)| < \varepsilon}$$

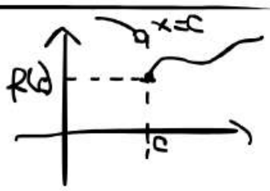
$h > 0 \quad h \in \mathbb{R}^+$ $\qquad$ $\boxed{h>0}$ $\quad h \in \mathbb{R}^+$

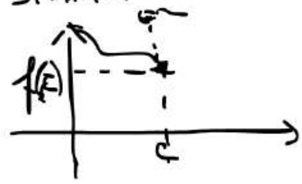
$c \quad c+h$ $\qquad\qquad$ $c-h \quad c$

DA DESTRA $\qquad\qquad$ DA SINISTRA

A $\boxed{\lim_{x \to c^+} f(x) = f(c)}$ CONTINUITÀ DELLA FUNZ. DALLA DESTRA

B $\boxed{\lim_{x \to c^-} f(x) = f(x)}$ CONTINUITÀ DELLA FUNZ. DALLA SINISTRA

A 

B 

$$\boxed{f(x) = \frac{2x^2-x-1}{x-1}}$$

Dom f: $\underbrace{x \neq 1}$

$$2x^2-x-1$$

$$\Delta = 1+8 = 9$$

$$x_{1,2} = \frac{1 \pm \sqrt{9}}{4} \nearrow \frac{1+3}{4} \quad \searrow \frac{1-3}{4}$$

$$x_1 = -\frac{\cancel{2}1}{\cancel{4}2} \quad x_2 = 1$$

$$\lim_{x \to 1} \frac{2x^2-x-1}{x-1} = \frac{(2x+1)\cancel{(x-1)}}{\cancel{(x-1)}}$$

$$\lim_{x \to 1} 2x+1 = 2 \cdot 1 + 1 = \underline{3}$$

(PUNTO DI DISCONT. DI 1^a SPECIE)
O ELIMINABILE

OPERAZIONI SUI LIMITI

Somma algebrica fra limiti

Il limite della somma di due funzioni è la somma dei limiti delle due, tenendo che entrambe possiedono limite finito.

$\lim_{x \to c} f_1(x) = l_1$

$\lim_{x \to c} f_2(x) = l_2$

$l_1, l_2 \in \mathbb{R}$

$$\boxed{\lim_{x \to c} f_1(x) \pm f_2(x) = \lim_{x \to c} f_1(x) \pm \lim_{x \to c} f_2(x)}$$

$\lim_{x \to c} f_1(x) = l_1 \qquad \forall \frac{\varepsilon}{2} > 0 \ \exists \ I_1(c) : \forall x \in I(c) \Rightarrow |f_1(x) - l_1| < \frac{\varepsilon}{2}$

$$\Downarrow$$

$$-\frac{\varepsilon}{2} < f_1(x) - l_1 < \frac{\varepsilon}{2}$$

$$\boxed{l_1 - \frac{\varepsilon}{2} < f_1(x) < l_1 + \frac{\varepsilon}{2}}$$

$\lim_{x \to c} f_2(x) = l_2 \qquad \forall \frac{\varepsilon}{2} > 0 \ \exists \ I_2(c) : \forall x \in I(c) \Rightarrow |f_2(x) - l_2| < \frac{\varepsilon}{2}$

$$-\frac{\varepsilon}{2} < f_2(x) - l_2 < \frac{\varepsilon}{2}$$

$$\boxed{l_2 - \frac{\varepsilon}{2} < f_2(x) < l_2 + \frac{\varepsilon}{2}}$$

$I = I_1(c) \cap I_2(c)$

$$\begin{cases} l_1 - \frac{\varepsilon}{2} < f_1(x) < l_1 + \frac{\varepsilon}{2} \\ l_2 - \frac{\varepsilon}{2} < f_2(x) < l_2 + \frac{\varepsilon}{2} \end{cases}$$

SOMMO MEMBRO A MEMBRO

$$l_1 + l_2 - \frac{\varepsilon}{2} - \frac{\varepsilon}{2} < f_1(x) + f_2(x) < l_1 + l_2 + \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$$

$$\boxed{(l_1 + l_2) - \varepsilon < f_1(x) + f_2(x) < (l_1 + l_2) + \varepsilon}$$

$$\Downarrow$$

$\forall \varepsilon > 0 \; \lim_{I = I_1 \cap I_2}$ $\quad |(f_1(x) + f_2(x)) - (l_1 + l_2)| < \varepsilon$ $\quad$ c.v.d

ES. $\lim_{x \to \pi} (x+5) = \lim_{x \to \pi} x + \lim_{x \to \pi} 5 = \pi + 5$

---

1) $\lim f_1(x) = l \qquad \lim f_2(x) = \pm\infty$

$\lim [f_1(x) + f_2(x)] = l \pm \infty = \pm\infty$

$\lim [f_1(x) - f_2(x)] = l \mp \infty = \mp\infty$

2) $\lim f_1(x) = +\infty \qquad \lim f_2(x) = +\infty$

$\lim [f_1(x) + f_2(x)] = +\infty + \infty = +\infty$

$\lim [f_1(x) - f_2(x)] = +\infty - \infty$?!? FORMA INDETERMINATA

3) $\lim f_1(x) = -\infty \qquad \lim f_2(x) = -\infty$

$\lim [f_1(x) + f_2(x)] = -\infty - \infty = -\infty$

$\lim [-f_1(x) - f_2(x)] = -\infty + \infty$?!! FORMA INDETERM.

4) $\lim f_1(x) = +\infty \qquad \lim f_2(x) = -\infty$

$\lim (f_1(x) - f_2(x)) = +\infty - (-\infty) = +\infty + \infty = +\infty$

$\lim (f_1(x) + f_2(x)) = +\infty + (-\infty) = +\infty - \infty$?!! FORMA INDETERM.

Prodotto di limiti di funzioni

A $\boxed{\lim_{x \to c} K \cdot f(x) = K \underbrace{\lim f(x)}_{x \to} = K \ell}$ $K \in \mathbb{R}$

$\boxed{\lim_{x \to c} f(x) = \ell}$

Il limite del prodotto di una costante per una funzione risulta essere uguale al prodotto della costante per il valore del limite della funzione.

$\forall \varepsilon > 0 \; \exists I(c) \ldots \Rightarrow |K \cdot f(x) - K\ell| < \varepsilon$

$|K \cdot (f(x) - \ell)| < \varepsilon$

$|K| \, |f(x) - \ell| < \varepsilon$

ES. $\lim_{x \to 2} 4 \cdot x = 4 \lim_{x \to 2} x = 4 \cdot 2 = 8$

$\to K \neq 0$ $\quad |f(x) - \ell| < \frac{\varepsilon}{|K|}$

(P3) $\lim_{x\to c} f_1(x) = l_1 \qquad \lim_{x\to c} f_2(x) = l_2$

$$\lim_{x\to c}\left[f_1(x)\cdot f_2(x)\right] = l_1 \cdot l_2$$

CASO $\quad l_1 = l_2 = 0$

$\forall \varepsilon > 0 \quad \exists\, I_1(c) : \forall x \in I_1(c) \Longrightarrow |f_1(x)| < \varepsilon$

$\forall \varepsilon > 0 \quad \exists\, I_2(c) : \forall x \in I_2(c) \Longrightarrow |f_2(x)| < \varepsilon$

$I = I_1 \cap I_2$

$I = I_1 \cap I_2$

$$\begin{cases} |f_1(x)| < \varepsilon \\ |f_2(x)| < \varepsilon \end{cases}$$

MOLTIP. MEMBRO A MEMBRO

$$|f_1(x)| \cdot |f_2(x)| < \varepsilon^2$$

$\varepsilon^2 < \varepsilon$

$$\underline{|f_1(x) f_2(x)| < \varepsilon^2} \Longrightarrow \boxed{\lim_{x \to c} f_1 f_2(x) = 0}$$

$l_1 \neq 0 \qquad l_2 \neq 0$

$$\lim_{x \to c} f_1(x) = l_1 \implies \lim_{x \to c} [f_1(x) - l_1] = 0$$

$$\lim_{x \to c} f_2(x) = l_2 \implies \lim_{x \to c} [f_2(x) - l_2] = 0$$

$$\lim_{x \to c} \{\underbrace{[f_1(x) - l_1][f_2(x) - l_2]}_{\varphi}\} = 0 \qquad \lim_{x \to c} \varphi(x) = 0$$

$$\varphi(x) = [f_1(x) - l_1][f_2(x) - l_2]$$

$$\varphi(x) = f_1(x) f_2(x) - l_2 f_1(x) - l_1 f_2(x) + l_1 l_2$$

$$\lim_{x \to c} \left[f_1(x) f_2(x) - \underline{l_2 f_1(x)} - l_1 f_2(x) + l_1 l_2 \right] =$$

$$= \underline{\lim_{x \to c} f_1(x) f_2(x)} - l_2 \underbrace{\lim_{x \to c} f_1(x)}_{l_1} - l_1 \underbrace{\lim_{x \to c} f_2(x)}_{l_2} + l_1 l_2 = 0$$

$$\boxed{\lim_{x \to c} f_1(x) f_2(x) = l_1 l_2} + \cancel{l_1 l_2} - \cancel{l_1 l_2}$$

c.v.d

$$\lim f_1(x) = \pm\infty \qquad \lim f_2(x) = \pm\infty$$

$$\lim [f_1(x) \cdot f_2(x)] = \pm\infty \cdot (\pm\infty) = \pm\infty$$

OK!

$$\lim_{x \to a} f_1(x) = \infty \qquad \lim f_2(x) = \infty$$

$$\lim [f_1(x)\, f_2(x)] = \boxed{\infty \cdot \infty}$$

?!?

FORMA INDETERM.

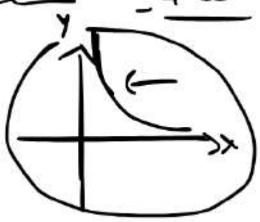
$$\lim f_1(x) = 0 \qquad \lim f_2(x) = \infty$$

$$\lim [f_1(x)\, f_2(x)] = \boxed{0 \cdot \infty}$$

?!?

FORMA INDETERM.

$$\lim_{x \to 0^+} \frac{1}{x^2} = \lim_{x \to 0^+} \frac{1}{x} \cdot \frac{1}{x} = \lim_{x \to 0^+} \frac{1}{x} \cdot \lim_{x \to 0^+} \frac{1}{x} = +\infty \cdot (+\infty)$$
$$= +\infty$$



Limite della potenza di una funzione

$$\boxed{\lim [f(x)]^n = [\lim f(x)]^n}$$

$n \in \mathbb{N}$

Il limite della potenza ennesima di una funzione corrisponde alla potenza ennesima del limite della funzione stessa

$\lim f(x) = +\infty$ $\quad \lim [f(x)]^n = +\infty$

$\lim f(x) = -\infty$ $\quad \lim [f(x)]^n \to -\infty$ È n DISPARI

$\to +\infty$ È n PARI

Limite del quoziente di due funzioni

(1) $\lim_{x \to c} f(x) = \ell \implies \lim_{x \to c} \frac{1}{f(x)} = \frac{1}{\ell}$ $\ell \in \mathbb{R} \setminus \{0\}$

$\forall \varepsilon > 0 \ \exists I(c) : \forall x \in I(c) \implies |f(x) - \ell| < \varepsilon$

FISSATO $\varepsilon = \frac{|\ell|}{2}$ DAL TEOREMA DELLA PERMANENZA DEL SEGNO

$\forall x \in I_1(c)$

$\boxed{|f(x) - l| < \varepsilon}$

$\varepsilon = \frac{|l|}{2}$

$-\varepsilon < f(x) - l < \varepsilon$

$l - \frac{|l|}{2} < f(x) < l + \frac{|l|}{2} \rightarrow \begin{cases} f(x) < l + \frac{|l|}{2} \\ f(x) > l - \frac{|l|}{2} \end{cases}$

$l > 0 \quad f(x) > l - \frac{l}{2} \Longrightarrow f(x) > \frac{l}{2} = \frac{|l|}{2}$

$\boxed{|f(x)| > \frac{|l|}{2}} \quad \forall x \in I_2(c)$

$I = I_1 \cap I_2$

$$\begin{cases} |f(x) - l| < \varepsilon \Rightarrow -\varepsilon < f(x) - l < \varepsilon \\ |f(x)| > \frac{l}{2} \Longrightarrow f(x) < -\frac{l}{2} \vee f(x) > \frac{l}{2} \end{cases}$$

$$\lim_{x \to c} \frac{1}{f(x)} = \frac{1}{l} \Rightarrow \left| \frac{1}{f(x)} - \frac{1}{l} \right|$$