LEZIONE 12

$\ell \neq 0$

Limite del quoziente di due funzioni

$$\lim_{x \to c} f(x) = \ell \implies \lim_{x \to c} \frac{1}{f(x)} = \frac{1}{\ell}$$

Dim

$$\forall \varepsilon > 0 \ \exists I(c): \forall x \in I(c) \implies |f(x) - \ell| < \varepsilon$$

$$\varepsilon = \frac{|\ell|}{2}$$

PERMANENZA DEL SEGNO

$\varepsilon = \frac{|\ell|}{2}$ $\qquad \lim\limits_{x \to c} f(x) = \ell$

$|f(x) - \ell| < \varepsilon \Longrightarrow |f(x) - \ell| < \frac{|\ell|}{2}$

$-\frac{|\ell|}{2} < f(x) - \ell < \frac{|\ell|}{2} \Longrightarrow \begin{cases} f(x) - \ell < \frac{|\ell|}{2} \\ f(x) - \ell > -\frac{|\ell|}{2} \end{cases}$

$\begin{cases} f(x) < \ell + \frac{|\ell|}{2} \\ f(x) > \ell - \frac{|\ell|}{2} \end{cases}$ se $\ell > 0 \Longrightarrow \begin{cases} f(x) < \ell + \frac{\ell}{2} = \frac{3}{2}\ell \\ \boxed{f(x) > \frac{\ell}{2}} \end{cases}$

$$\begin{cases} f(x) < l - \frac{l}{2} \Rightarrow f(x) < \frac{l}{2} \vee \\ \cancel{f(x) > +l + \frac{l}{2} \Rightarrow f(x) > \frac{3}{2} l} \end{cases}$$

$\underline{l < 0}$

$f(x) > \frac{l}{2}$ $\qquad$ $f(x) < \frac{l}{2}$

[redacted]
IN UNIONE

$l > 0$ $\qquad$ $l < 0$

$$\boxed{f(x) > \frac{|l|}{2} \qquad f(x) < -\frac{|l|}{2}}$$

$$\boxed{f(x) < -\frac{|l|}{2} \lor f(x) > \frac{|l|}{2} \Rightarrow}$$

$$\boxed{|f(x)| > \frac{|l|}{2}}$$

$\lim_{x \to e} f(x) = l$

$$\begin{array}{l} I_1 \\ I_2 \end{array} \boxed{\begin{cases} |f(x) - l| < \varepsilon \\ |f(x)| \geq \frac{|l|}{2} \end{cases}}$$

$I = I_1 \cap I_2$

$$\boxed{\lim_{x \to c} \frac{1}{f(x)} = \frac{1}{l}}$$

$$\boxed{\left|\frac{1}{f(x)} - \frac{1}{l}\right| = \left|\frac{l - f(x)}{l \cdot f(x)}\right| = \frac{|\cancel{l} - f(x)|}{|l| \cdot |f(x)|}}$$

$N \quad |l - f(x)| = |f(x) - l| < \varepsilon$

$$|f(x)| > \frac{|l|}{2} \implies \frac{1}{|f(x)|} < \frac{1}{\frac{|l|}{2}} \qquad \varepsilon = 0{,}1$$

$$\left|\frac{1}{f(x)} - \frac{1}{l}\right| = \frac{|l - f(x)|}{|l| \cdot f(x)} < \frac{\varepsilon}{|l| \frac{|l|}{2}} = \frac{2\varepsilon}{l^2}$$

INFINIT

$$\left|\frac{1}{f(x)} - \frac{1}{l}\right| < \frac{2\varepsilon}{l^2} \qquad \lim_{x \to c} \frac{1}{f(x)} = \frac{1}{l}$$

c.v.d

$$\lim \frac{f_1(x)}{f_2(x)} = \frac{\lim f_1(x)}{\lim f_2(x)}$$

$$\frac{f_1(x)}{f_2(x)} = f_1(x) \cdot \frac{1}{f_2(x)}$$

$$\lim \frac{f_1(x)}{f_2(x)} = \lim f_1(x) \cdot \lim \frac{1}{f_2(x)}$$

$$l_1 \cdot \frac{1}{l_2} = \frac{l_1}{l_2}$$

1) $\lim f_1(x) = l \neq 0 \qquad \lim f_2(x) = 0$

$$\lim \frac{f_1(x)}{f_2(x)} = \frac{l}{0} = \infty$$

2) $\lim f_1(x) = \infty \qquad \lim f_2(x) = l$

$$\lim \frac{f_1(x)}{f_2(x)} = \frac{\infty}{l} = \infty$$

anche con $l = 0$

3) $\lim f_1(x) = l \qquad \lim f_2(x) = \infty$

$$\lim \frac{f_1(x)}{f_2(x)} = \frac{l}{\infty} = 0$$

anche con $l = 0$

4) $\lim f_1(x) = \infty$ $\lim f_2(x) = \infty$

$$\lim \frac{f_1(x)}{f_2(x)} = \left[\frac{\infty}{\infty}\right] \quad ?!?$$ FORMA INDETERMINATA

$\lim f_1(x) = 0$ $\lim f_2(x) = 0$

$$\lim \frac{f_1(x)}{f_2(x)} = \left[\frac{0}{0}\right] \quad ?!?$$ FORMA INDETERMINATA

5) $\lim f(x) = 0$ $\lim \frac{1}{f(x)} = \frac{1}{0} = \infty$

6) $\lim f(x) = \infty$ $\lim \frac{1}{f(x)} = \frac{1}{\infty} = 0$

Limiti delle funzioni razionali intere

$$f(x) = a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_n \quad \boxed{a_0 \neq 0}$$

$$\lim_{x \to c} f(x) = \lim_{x \to c} (a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_n) = f(c)$$

Le funzioni razionali intere sono sempre funzioni continue in tutto il loro dominio

$$\lim_{x \to \pm\infty} (a_0 x^n + a_1 x^{n-1} + a_2 x^{n-2} + \dots + a_n) =$$

$$= \lim_{x \to \pm\infty} \left[x^n \left(a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \dots + \frac{a_n}{x^n} \right) \right]$$

$$\lim_{x \to \pm\infty} x^n \cdot \lim_{x \to \pm\infty} \left(a_0 + \underset{\downarrow \atop 0}{\frac{a_1}{x}} + \underset{\downarrow \atop 0}{\frac{a_2}{x^2}} + \cdots + \underset{\downarrow \atop 0}{\frac{a_n}{x^n}} \right) =$$

$$= a_0 \cdot \lim_{x \to \pm\infty} x^n$$

Tutte le funzioni razionali intere hanno per limite o + o - infinito.

se n pos. $+\infty$

se n NEGATIVO

se $x \to -\infty \Rightarrow \lim_{x \to -\infty} x^n = -\infty$

se $x \to +\infty \Rightarrow \lim_{x \to +\infty} x^n = +\infty$

$$\lim_{x \to +\infty} (-2x^3 + 3x - 5) =$$

$$= \lim_{x \to +\infty} \left[x^3 \left(-2 + \underset{\substack{\downarrow \\ 0}}{\frac{3}{x^2}} - \underset{\substack{\downarrow \\ 0}}{\frac{5}{x^3}} \right) \right] = -2 \lim_{x \to +\infty} x^3 = -2 \cdot (+\infty)^3 =$$

$$= -2 \cdot (+\infty) = \underline{\underline{-\infty}}$$

Limiti delle funzioni razionali fratte

A $\boxed{\lim_{x \to c} f(x)}$ $c \in \mathrm{Dom}f$

$$f(x) = \frac{P(x)}{Q(x)} = \frac{a_0 x^m + a_1 x^{m-1} + \cdots + a_{m-1}x + a_m}{b_0 x^n + b_1 x^{n-1} + \cdots + b_{n-1}x + b_n}$$

a)

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)} = f(c)$$

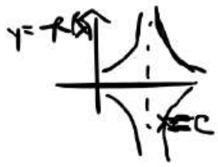
IN QUESTO CASO C'È CONTINUITÀ IN $x = c$

$c \in \mathrm{Dom}f : Q(c) \neq 0$

$\exists$ VALORE FINITO

b) $c \notin \mathrm{Dom}\, f : Q(c) = 0$

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{0} = \infty$$



c) $Q(c) = P(c) = 0$

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)} = \left[\frac{0}{0}\right]$$

FORMA INDETERM.

[SI CERCANO STRADE ALGEBRICHE]

(a) $\lim_{x \to 1} \frac{x^3+x-5}{2x^2+1} = \frac{1^3+1-5}{2 \cdot 1^2+1} = \frac{1+1-5}{2+1} = \frac{-\cancel{2}}{\cancel{3}} = -1 = \cancel{(}-1)$

(b) $\lim_{x \to -2} \frac{x^2+4}{x^2-4} = \frac{8}{0} = \infty$ → RUFFINI

(c) $\lim_{x \to 3} \frac{x^3-3x^2+x-3}{x^2-2x-3} = \frac{\cancel{27}-3\cdot\cancel{9}+\cancel{3}-\cancel{3}}{\cancel{9}-\cancel{6}-\cancel{3}} = \left[\frac{0}{0}\right]$

→ S TRINOMIO SPECIALE

$\lim_{x \to 3} \frac{\cancel{(x-3)} \cdot (x^2+1)}{\cancel{(x-3)}\ (x+1)} = \frac{9+1}{3+1} = \frac{\cancel{10}}{\cancel{4}_2} = \frac{5}{2}$

$\boxed{B}$ $\lim_{x\to\infty} f(x) =$

$$\lim_{x\to\pm\infty} \frac{a_0x^m + a_1x^{m-1} + \dots + a_{m-1}x + a_m}{b_0x^n + b_1x^{n-1} + \dots + b_{n-1}x + b_n} =$$

$$= \lim_{x\to\pm\infty} \frac{x^m}{x^n} \frac{\left(a_0 + \frac{a_1}{x} + \dots + \frac{a_{m-1}}{x^{m-1}} + \frac{a_m}{x^m}\right)}{\left(b_0 + \frac{b_1}{x} + \dots + \frac{b_{n-1}}{x^{n-1}} + \frac{b_n}{x^n}\right)}$$

$$= \frac{a_0}{b_0} \cdot \lim_{x\to\pm\infty} \frac{x^m}{x^n}$$

1) $m > n$

$\lim_{x\to+\infty} x^{m-n} = +\infty$

$\lim_{x\to-\infty} x^{m-n}$ $\to +\infty$ $m-n$ PARI; $\to -\infty$ $m-n$ DISPARI

2) $m = n$

$\lim_{x\to+\infty} x^{m-n} = \lim_{x\to\pm\infty} x^0 = 1$

$\lim_{x\to+\infty} f(x) = \frac{a_0}{b_0}$

3) $m < n$

$\lim_{x\to\pm\infty} x^{m-n} = \lim_{x\to\pm\infty} \frac{1}{x^{n-m}} =$

$\frac{1}{\pm\infty} = 0$

CASO $m > m$

$$\lim_{x \to +\infty} \frac{2x^3+1}{-3x^2+1} = \lim_{x \to +\infty} \frac{\cancel{x^3}\left(2+\frac{1}{x^3}\right)^{\to 0}}{\cancel{x^2}\left(-3+\frac{1}{x^2}\right)_{\to 0}} = -\frac{2}{3} \lim_{x \to +\infty} x$$

$$= -\frac{2}{3} \cdot (+\infty) = \underline{-\infty}$$

CASO $m = m$

$$\lim_{x \to \pm\infty} \frac{5x^3-3x^2+2x-1}{8x^3+3} = \lim_{x \to \pm\infty} \frac{\cancel{x^3}\left(5-\frac{3}{x}+\frac{2}{x^2}-\frac{1}{x^3}\right)}{\cancel{x^3}\left(8+\frac{3}{x^3}\right)} =$$

$$= \frac{5}{8}$$

$m < n$

$$\lim_{x \to -\infty} \frac{2x-3}{x^4+1} = \lim_{x \to -\infty} \frac{\cancel{x}(2 - \underbrace{\tfrac{3}{x}}_{\to 0})}{x^{\cancel{4}\,3}(1 + \underbrace{\tfrac{1}{x^4}}_{\to 0})} = 2 \cdot \lim_{x \to -\infty} \frac{1}{x^3} =$$

$$= 2 \cdot \frac{1}{(-\infty)^3} =$$

$$= 2 \cdot \underbrace{\left(-\frac{1}{\infty}\right)}_{\to 0} = \underline{0}$$

LIMITI NOTEVOLI

$$\boxed{\lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{x} = e}$$

$$\boxed{\lim_{z \to 0} (1+z)^{\frac{1}{z}} = e}$$

$$\boxed{\lim_{x \to 0} (1+x)^{\frac{1}{x}} = e}$$

$$\boxed{z = \frac{1}{x} \quad x = \frac{1}{z}}$$

$\left(1 + \frac{1}{1}\right)^{1} = (1+1)^{1} = 2^{1}$

$\left(1 + \frac{1}{2}\right)^{2} = \left(\frac{3}{2}\right)^{2} = \frac{9}{4} = 2{,}25$

$\left(1 + \frac{1}{3}\right)^{3} = \left(\frac{4}{3}\right)^{3} = \frac{64}{27} \simeq 2{,}3$

$\vdots$

$\left(1 + \frac{1}{100}\right)^{100} = \left(\frac{101}{100}\right)^{100}$

$$\lim_{x\to\infty}\left(1-\frac{2}{x}\right)^{x} = \lim_{z\to\infty}\left(1+\frac{1}{z}\right)^{-2z} =$$

$$-\frac{2}{x} = \frac{1}{z}$$

$$= \boxed{\lim_{z\to\infty}\left[\left(1+\frac{1}{z}\right)^{z}\right]^{-2}} = \underline{e^{-2}} = \frac{1}{e^{2}}$$

$$\to e$$

$$-\frac{x}{2} = z$$

$$x = -2z$$

$$\boxed{\lim_{x \to 0} \frac{\log_a (1+x)}{x} = \log_a e} \qquad a \in \mathbb{R}^+ - \{1\}$$

$$\lim_{x \to 0} \frac{\log_a (1+0)}{0} = \frac{\log_a (1+0)}{0} = \frac{\log_a 1}{0} = \left[\frac{0}{0}\right]$$

$$\lim_{x \to 0} \frac{\log_a (1+x)}{x} = \lim_{x \to 0} \frac{1}{x} \cdot \log_a (1+x) = \lim_{x \to 0} \log_a (1+x)^{\frac{1}{x}} =$$

$$= \log_a \left[\lim_{x \to 0} (1+x)^{\frac{1}{x}}\right] = \boxed{\log_a e}$$

$$\boxed{a \log b = \log b^a}$$

$$a = e \quad \left| \lim_{x \to 0} \frac{\log_e (1+x)}{x} = \log_e e = 1 \right|$$