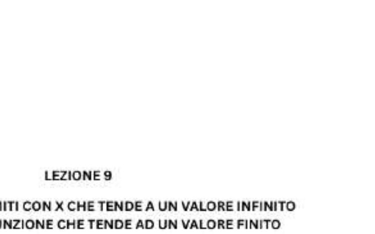
LEZIONE 9

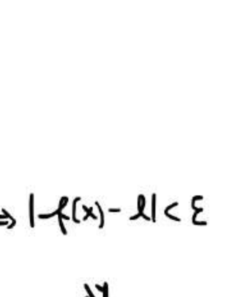
RECAP SUI LIMITI CON X CHE TENDE A UN VALORE INFINITO E CON LA FUNZIONE CHE TENDE AD UN VALORE FINITO

$\lim_{x\to\infty} f(x) = l$; $\forall \varepsilon > 0 \ \exists\, I(\infty) : \forall x \in I(\infty) \Rightarrow |f(x) - l| < \varepsilon$

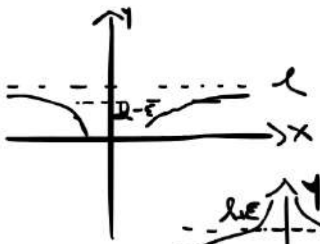
a) $\lim_{x\to+\infty} f(x) = l^-$



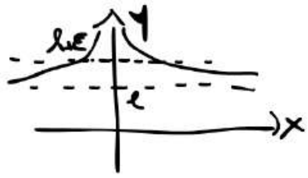
b) $\lim_{x\to+\infty} f(x) = l^+$



$$\lim_{x \to \infty} f(x) = l^-$$



$$\lim_{x \to \infty} f(x) = l^+$$



ESEMPIO Dom f $\mathbb{R} \setminus \{0\}$

$$\lim_{x \to \infty} \frac{x^2+1}{x^2} = 1^+$$

$$\forall \varepsilon > 0 \ \exists \ I(\infty) : \forall x \in I(\infty) \Longrightarrow 1 \leq \frac{x^2+1}{x^2} \leq 1 + \varepsilon$$

$$0 \leq \frac{x^2+1}{x^2} - 1 \leq \varepsilon \Rightarrow 0 \leq \frac{\cancel{x^2}+1-\cancel{x^2}}{x^2} \leq \varepsilon \qquad x = \pm\sqrt{\frac{1}{\varepsilon}}$$

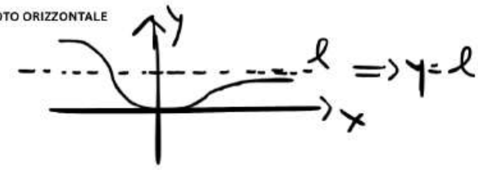
$$\left(0 \leq \frac{1}{x^2} \leq \varepsilon\right) \Rightarrow \begin{cases} \frac{1}{x^2} \geq 0 \quad \forall x \in \text{Dom} f \\ \frac{1}{x^2} \leq \varepsilon \underset{\text{passo inversi}}{\Longrightarrow} \boxed{x^2 \geq \frac{1}{\varepsilon}} \end{cases} \quad \boxed{x \leq -\frac{1}{\sqrt{\varepsilon}} \vee x \geq \frac{1}{\sqrt{\varepsilon}}}$$

$$I(\infty) = \left]-\infty, -\frac{1}{\sqrt{\varepsilon}}\right] \cup \left[\frac{1}{\sqrt{\varepsilon}}, +\infty\right[$$

$$\boxed{I(-\infty) \cup I(+\infty)}$$

ASINTOTO ORIZZONTALE

$$\lim_{x \to +\infty} f(x) = l$$

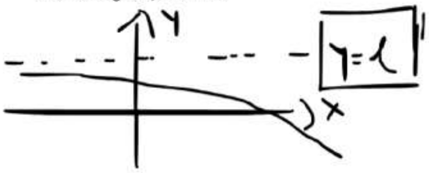


$$\boxed{y = l}$$

Asintoto orizzontale di f(x)

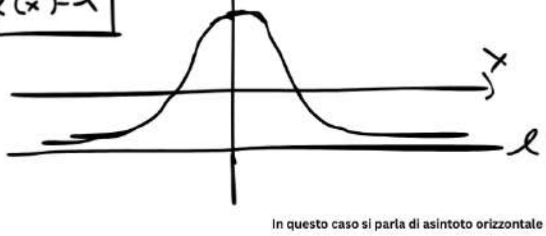
in questo caso la retta y=l è un asintoto orizzontale destro perché la x si avvicina a valori molto grandi, a + infinito.

$$\lim_{x \to -\infty} f(x) = l$$



In questo caso specifico, invece x tende ad un valore molto piccolo, cioè meno infinito e quindi questo è un asintoto orizzontale sinistro.

$$\boxed{\lim_{x \to \infty} f(x) = l}$$



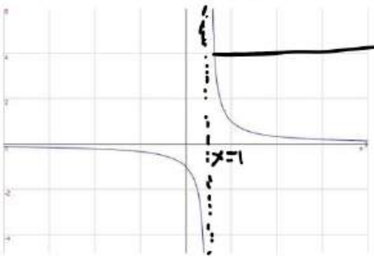
In questo caso si parla di asintoto orizzontale

LIMITE PER UNA FUNZIONE CHE TENDE AD UN VALORE INFINITO PER X CHE TENDE AD UN VALORE FINITO

$$\text{Dom} f : \mathbb{R} \setminus \{1\}$$

$$y = f(x) = \frac{1}{x-1}$$

Essendo x=1, il punto dove la funzione non è definita, vogliamo vedere come si comporta la funzione f(x), quando x si avvicina al valore 1.



ASINTOTO VERTICALE

x	$f(x)$
0,9	-10
0,99	-100
0,999	-1000
1,1	10
1,01	100
1,001	1000

DESTRO

$$\lim_{x \to c^+} f(x) = \infty$$

SINISTRO

$$\lim_{x \to c^-} f(x) = \infty$$

In base a quanto visto precedentemente si ottiene:

$$\lim_{x \to 1} \frac{1}{x-1} = \infty$$

$$\lim_{x \to 1^-} \frac{1}{x-1} = -\infty$$

$$\lim_{x \to 1^+} \frac{1}{x-1} = +\infty$$

$$\left|\frac{1}{x-1}\right| > M \Longrightarrow \boxed{\frac{1}{x-1} < -M \ \vee \ \frac{1}{x-1} > M}$$

M valore arbitr grande

$\downarrow$ PASS. AGLI INVERSI

$$|x-1| < \frac{1}{M} \Longrightarrow -\frac{1}{M} < x-1 < \frac{1}{M} \Longrightarrow \boxed{1-\frac{1}{M} < x < 1+\frac{1}{M}}$$

$$1-\frac{1}{M}<x<1+\frac{1}{M} \Longrightarrow \left]1-\underbrace{\frac{1}{M}}\,;\,1+\underbrace{\frac{1}{M}}\right[$$

POSSIBILI VALORI DI M

$M = 10$

$M = 100$

$M = 1000$

$\frac{1}{}$

Da tutte queste considerazione viene fuori la definizione di limite per x che tende ad un valore finito con la funzione che tende ad un valore infinito.

$$\boxed{\forall M>0\ \exists I(c) : \forall x \in I(c) \Rightarrow |f(x)|>M}$$

$$\lim_{x \to c} f(x) = \infty$$

ESEMPIO

$$\lim_{x \to 1} \frac{2x-1}{x-1} = \infty \qquad \text{Dom } f: \mathbb{R} \setminus \{1\}$$

$$\forall M > 0 \ \exists I(1) \setminus \{1\} : \forall x \in I(1) \Rightarrow \left| \frac{2x-1}{x-1} \right| > M$$

$$\boxed{\frac{2x-1}{x-1} < -M \ (A) \quad \vee \quad \frac{2x-1}{x-1} > M \ (B)}$$

(A) $\dfrac{2x-1}{x-1} < -M \Rightarrow \dfrac{2x-1}{x-1} + M < 0$

$\dfrac{2x-1+M(x-1)}{x-1} < 0$

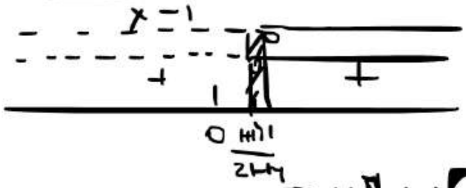
N $2x-1+Mx-M>0$

$(2+M)x > 1+M$

$\boxed{x > \dfrac{1+M}{2+M}}$

D $x-1>0$

$\boxed{x>1}$



$\dfrac{1+M}{2+M} < x < 1 \qquad I^-(1) = \left]\dfrac{1+M}{2+M}, 1\right[$

$$\frac{2x-1}{x-1} > M \Longrightarrow \frac{2x-1}{x-1} - M > 0$$

$$\frac{2x-1-M(x-1)}{x-1} > 0$$

N

$2x-1-Mx+M>0$

$(2-M)x > 1-M$

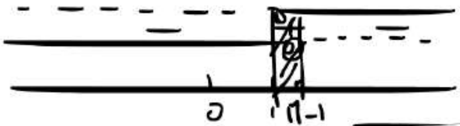
$(M-2)x < M-1$

$x < \frac{M-1}{M-2}$

D

$x-1>0$

$x>1$



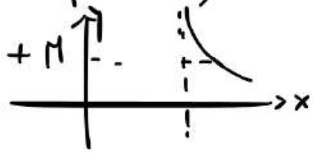
$$I^+(1) = \left]1; \frac{M-1}{M-2}\right[\qquad \boxed{1<x<\frac{M-1}{M-2}}$$

$$\mathrm{I}(1)\setminus\{1\} = \left]\frac{1+M}{2+M} ; \frac{M-1}{M-2}\right[$$

$M=10$ $$\left]\frac{11}{12} ; \frac{9}{8}\right[\setminus \{1\}$$

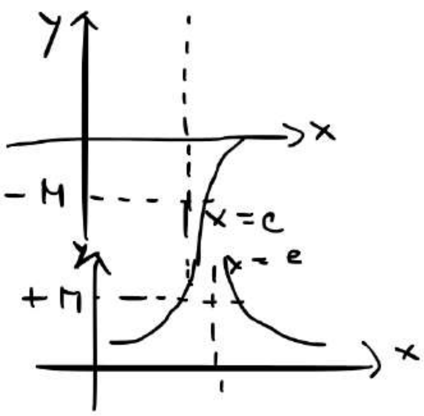
$\lim_{x \to c^-} f(x) = +\infty$

$\lim_{x \to e^+} f(x) = +\infty$



$\lim_{x \to c^+} f(x) = -\infty$

$\lim_{x \to c} f(x) = +\infty$



1

$$\lim_{x \to 2} \frac{1}{4x - x^2 - 4} = -\infty$$

$$\forall M > 0 \ \exists\, I(2) \setminus \{2\} : \forall x \in I(2) \Longrightarrow \frac{1}{4x - x^2 - 4} < -M$$

$$\frac{1}{-(x^2 - 4x + 4)} < -M \Longrightarrow -\frac{1}{(x-2)^2} < -M \qquad X = (x-2)$$

$$\frac{1}{(x-2)^2} > M \Longrightarrow \underbrace{(x-2)}_{X}{}^2 < \frac{1}{M} \Longrightarrow X^2 < \frac{1}{M}$$

$$-\frac{1}{\sqrt{M}} < X < \frac{1}{\sqrt{M}} \Longrightarrow$$

$$-\frac{1}{\sqrt{M}} < X < \frac{1}{\sqrt{M}}$$

$$X = x - 2$$

$$M = 100$$

$$-\frac{1}{\sqrt{M}} < x - 2 < \frac{1}{\sqrt{M}}$$

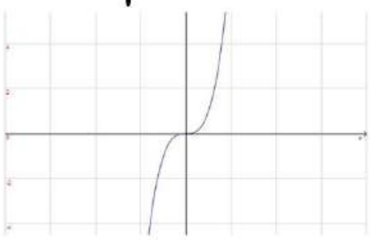
$$2 - 0{,}1 < x < 2 + 0{,}1$$

$$2 - \frac{1}{\sqrt{M}} < x < 2 + \frac{1}{\sqrt{M}}$$

$$I(2) \setminus \{2\} = \left]2 - \frac{1}{\sqrt{M}}, 2 + \frac{1}{\sqrt{M}}\right[$$

LIMITE PER X CHE TENDE ALL'INFINITO CON f(x) CHE TENDE ALL'INFINITO.

$y = x^3$



$$\boxed{\lim_{x \to \infty} f(x) = \infty}$$

x	$y = f(x)$
-10^4	-10^{12}
-10^3	-10^9
-10^2	-10^6
-10	-10^3
10	10^3
10^2	10^6
10^3	10^9
10^4	10^{12}

$$\lim_{x \to \infty} f(x) = \infty$$

$$\boxed{\forall M > 0 \ \exists I(\infty) : \forall x \in I(\infty) \Longrightarrow |f(x)| > M}$$

$$\lim_{x \to \infty} x^3 = \infty$$

$$\forall M > 0 \ \exists I(\infty) : \forall x \in I(\infty) \Longrightarrow |x^3| > M$$

$$\sqrt[3]{-8} = -\sqrt[3]{8}$$

$$x^3 < -M \quad \vee \quad x^3 > M$$

$$x < -\sqrt[3]{M} \quad \vee \quad x > \sqrt[3]{M}$$

$$I(\infty) = \left]-\infty ; -\sqrt[3]{M}\right[\ \cup \ \left]\sqrt[3]{M} ; +\infty\right[$$