LEZIONE 9

Esercizi su espressioni goniometriche con formule di addizione e sottrazione

$$\boxed{\frac{\sqrt{2}\cos\left(\frac{3}{4}\pi+x\right)}{\cos\left(x+\frac{2}{3}\pi\right)+\cos\left(x-\frac{2}{3}\pi\right)}}$$

$\frac{3}{4}\pi = \frac{3}{\cancel{4}_{1}}\cdot\cancel{180}^{45} = 135^\circ$

$\frac{2}{3}\pi = 120^\circ$

$\cos(135^\circ+x) = \underline{\cos 135^\circ}\cos x - \underline{\operatorname{sen} 135^\circ}\operatorname{sen} x =$

$= \underline{\cos(90^\circ+45^\circ)}\cos x - \operatorname{sen}(90^\circ+45^\circ)\operatorname{sen} x = \underline{-\cos 45^\circ}\cos x - \operatorname{sen} 45^\circ \operatorname{sen} x$

$$\boxed{\cos(135^\circ + x) = -\frac{\sqrt{2}}{2}\cos x - \frac{\sqrt{2}}{2}\operatorname{sen} x}$$

$$\cos(x + 120^\circ) = \cos x \cos 120^\circ - \operatorname{sen} x \operatorname{sen} 120^\circ =$$

$$= \cos x \cos(90^\circ + 30^\circ) - \operatorname{sen} x \operatorname{sen}(90^\circ + 30^\circ) =$$

$$= \cos x \cdot (-\operatorname{sen} 30^\circ) - \operatorname{sen} x \cos 30^\circ =$$

$$= -\frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\operatorname{sen} x$$

$$\boxed{\cos(x - 120^\circ) = -\frac{1}{2}\cos x + \frac{\sqrt{3}}{2}\operatorname{sen} x}$$

$$\boxed{\cos(x + 120^\circ) = -\frac{1}{2}\cos x - \frac{\sqrt{3}}{2}\operatorname{sen} x}$$

$$\frac{\left(-\frac{\sqrt{2}}{2}\cos x-\frac{\sqrt{2}}{2}\operatorname{sen}x\right)\cdot\sqrt{2}}{-\frac{1}{2}\cos x-\cancel{\frac{\sqrt{3}}{2}\operatorname{sen}x}-\frac{1}{2}\cos x+\cancel{\frac{\sqrt{3}}{2}\operatorname{sen}x}}=\frac{-\cos x-\operatorname{sen}x}{-\cos x}=$$

$$=-\frac{\cos x-\operatorname{sen}x}{-\cos x}=\frac{\cancel{-}(\cos x+\operatorname{sen}x)}{\cancel{-}\cos x}=\frac{\cos x+\operatorname{sen}x}{\cos x}=$$

$$=\frac{\cancel{\cos x}}{\cancel{\cos x}}+\frac{\operatorname{sen}x}{\cos x}=\underline{1+\operatorname{tg}x}$$

$$\operatorname{sen}\alpha + \cos\left(\alpha + \frac{\pi}{6}\right) + \cos\left(\alpha + \frac{5}{6}\pi\right) = \qquad \frac{\pi}{6} = 30^\circ$$

$$= \operatorname{sen}\alpha + \boxed{\cos(\alpha + 30^\circ)} + \boxed{\cos(\alpha + 150^\circ)} = \qquad \frac{5}{6}\pi = 150^\circ$$

$$= \operatorname{sen}\alpha + \cos\alpha\cos 30^\circ - \operatorname{sen}\alpha\operatorname{sen}30^\circ + \cos\alpha\cos 150^\circ - \operatorname{sen}\alpha\operatorname{sen}150^\circ =$$

$$= \operatorname{sen}\alpha + \frac{\sqrt{3}}{2}\cos\alpha - \frac{1}{2}\operatorname{sen}\alpha + \cos\alpha\cos(180^\circ - 30^\circ) - \operatorname{sen}\alpha\operatorname{sen}(180^\circ - 30^\circ) =$$

$$= \operatorname{sen}\alpha + \frac{\sqrt{3}}{2}\cos\alpha - \frac{1}{2}\operatorname{sen}\alpha - \cos 30^\circ\cos\alpha - \operatorname{sen}30^\circ\operatorname{sen}\alpha =$$

$$= \cancel{\operatorname{sen}\alpha} + \cancel{\frac{\sqrt{3}}{2}\cos\alpha} - \cancel{\frac{1}{2}\operatorname{sen}\alpha} - \cancel{\frac{\sqrt{3}}{2}\cos\alpha} - \cancel{\frac{1}{2}\operatorname{sen}\alpha}$$

$$= 0$$

$$\frac{\operatorname{sen}(\alpha+\beta)}{\operatorname{sen}(\alpha-\beta)} - \frac{\operatorname{tg}(\alpha+\beta)}{\operatorname{tg}(\alpha-\beta)} =$$

$$= \frac{\operatorname{sen}\alpha\cos\beta + \operatorname{sen}\beta\cos\alpha}{\operatorname{sen}\alpha\cos\beta - \operatorname{sen}\beta\cos\alpha} - \frac{\dfrac{\operatorname{tg}\alpha + \operatorname{tg}\beta}{1 - \operatorname{tg}\alpha\operatorname{tg}\beta}}{\dfrac{\operatorname{tg}\alpha - \operatorname{tg}\beta}{1 + \operatorname{tg}\alpha\operatorname{tg}\beta}} =$$

$$= \frac{\operatorname{sen}\alpha\cos\beta + \operatorname{sen}\beta\cos\alpha}{\operatorname{sen}\alpha\cos\beta - \operatorname{sen}\beta\cos\alpha} - \frac{\dfrac{\dfrac{\operatorname{sen}\alpha}{\cos\alpha} + \dfrac{\operatorname{sen}\beta}{\cos\beta}}{1 - \dfrac{\operatorname{sen}\alpha}{\cos\alpha}\cdot\dfrac{\operatorname{sen}\beta}{\cos\beta}}}{\dfrac{\dfrac{\operatorname{sen}\alpha}{\cos\alpha} - \dfrac{\operatorname{sen}\beta}{\cos\beta}}{1 + \dfrac{\operatorname{sen}\alpha}{\cos\alpha}\cdot\dfrac{\operatorname{sen}\beta}{\cos\beta}}} =$$

$$= \frac{\operatorname{sen}\alpha\cos\beta + \operatorname{sen}\beta\cos\alpha}{\operatorname{sen}\alpha\cos\beta - \operatorname{sen}\beta\cos\alpha} - \frac{\dfrac{\dfrac{\operatorname{sen}\alpha\cos\beta + \operatorname{sen}\beta\cos\alpha}{\cancel{\cos\alpha\cos\beta}}}{\dfrac{\cos\alpha\cos\beta - \operatorname{sen}\alpha\operatorname{sen}\beta}{\cancel{\cos\alpha\cos\beta}}}}{\dfrac{\dfrac{\operatorname{sen}\alpha\cos\beta - \operatorname{sen}\beta\cos\alpha}{\cancel{\cos\alpha\cos\beta}}}{\dfrac{\cos\alpha\cos\beta + \operatorname{sen}\alpha\operatorname{sen}\beta}{\cancel{\cos\alpha\cos\beta}}}}$$

$$\frac{\operatorname{sen}\left(\alpha+\frac{\pi}{3}\right)+\operatorname{sen}\left(\alpha-\frac{\pi}{3}\right)-2\operatorname{sen}\alpha}{\cos\left(\alpha+\frac{\pi}{3}\right)+\cos\left(\alpha-\frac{\pi}{3}\right)-2\cos\alpha} = \qquad \frac{\pi}{3}=60^\circ$$

$$= \frac{\operatorname{sen}\alpha\cos\frac{\pi}{3}+\cancel{\operatorname{sen}\frac{\pi}{3}\cos\alpha}+\operatorname{sen}\alpha\cos\frac{\pi}{3}\cancel{-\operatorname{sen}\frac{\pi}{3}\cos\alpha}-2\operatorname{sen}\alpha}{\cos\alpha\cos\frac{\pi}{3}\cancel{-\operatorname{sen}\alpha\operatorname{sen}\frac{\pi}{3}}+\cos\alpha\cos\frac{\pi}{3}\cancel{+\operatorname{sen}\alpha\operatorname{sen}\frac{\pi}{3}}-2\cos\alpha}$$

$$= \frac{2\operatorname{sen}\alpha\cos\frac{\pi}{3}-2\operatorname{sen}\alpha}{2\cos\alpha\cos\frac{\pi}{3}-2\cos\alpha} = \frac{\cancel{2}\operatorname{sen}\alpha\,\cancel{\left(\cos\frac{\pi}{3}-1\right)}}{\cancel{2}\cos\alpha\,\cancel{\left(\cos\frac{\pi}{3}-1\right)}}$$

$$= \frac{\operatorname{sen}\alpha}{\cos\alpha} = \boxed{\operatorname{tg}\alpha}$$

RECAP DELLE FORMULE DI DUPLICAZIONE

$$\operatorname{sen} 2\alpha = \operatorname{sen}(\alpha+\alpha) = \operatorname{sen}\alpha\cos\alpha + \operatorname{sen}\alpha\cos\alpha = \underline{2\operatorname{sen}\alpha\cos\alpha}$$

$$\cos 2\alpha = \cos(\alpha+\alpha) = \cos\alpha\cdot\cos\alpha - \operatorname{sen}\alpha\operatorname{sen}\alpha = \underline{\cos^2\alpha - \operatorname{sen}^2\alpha}$$

$$\operatorname{tg} 2\alpha = \operatorname{tg}(\alpha+\alpha) = \frac{\operatorname{tg}\alpha + \operatorname{tg}\alpha}{1 - \operatorname{tg}\alpha\operatorname{tg}\alpha} = \frac{2\operatorname{tg}\alpha}{1-\operatorname{tg}^2\alpha}$$

FORMULE PARAMETRICHE

Lo scopo di queste formule è di esprimere il seno e il coseno di un angolo x, in funzione della tangente dell'angolo x dimezzato (x/2)

$$\boxed{\begin{array}{ll} \text{sen}\,2\alpha = 2\,\text{sen}\,\alpha\cos\alpha & \cos 2\alpha = \cos^2\alpha - \text{sen}^2\alpha \\ \text{sen}\,\alpha = 2\,\text{sen}\,\frac{\alpha}{2}\cos\frac{\alpha}{2} & \cos\alpha = \cos^2\frac{\alpha}{2} - \text{sen}^2\frac{\alpha}{2} \end{array}}$$

$$\boxed{\cos^2\frac{\alpha}{2} + \text{sen}^2\frac{\alpha}{2} = 1}$$

Dividendo le relazioni ottenute sopra per la relazione fondamentale goniometrica non cambia nulla perchè sto dividendo per 1.

$$\operatorname{sen}\alpha = \frac{2\operatorname{sen}\frac{\alpha}{2}\cos\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}+\operatorname{sen}^2\frac{\alpha}{2}} \qquad \cos\alpha = \frac{\cos^2\frac{\alpha}{2}-\operatorname{sen}^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}+\operatorname{sen}^2\frac{\alpha}{2}}$$

DIVIDO TUTTO PER $\cos^2\frac{\alpha}{2}$ $\quad \frac{\alpha}{2} \neq \frac{\pi}{2}+k\pi$ $\quad \boxed{\alpha \neq \pi + 2k\pi}$

$$\operatorname{sen}\alpha = \frac{\dfrac{2\operatorname{sen}\frac{\alpha}{2}\cancel{\cos\frac{\alpha}{2}}}{\cos^{\cancel{2}}\frac{\alpha}{2}}}{\dfrac{\cos^2\frac{\alpha}{2}+\operatorname{sen}^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}}} = \frac{2\operatorname{tg}\frac{\alpha}{2}}{1+\operatorname{tg}^2\frac{\alpha}{2}}$$

$$\boxed{\cos\alpha = \frac{\frac{\cos^2\frac{\alpha}{2} - \sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}}}{\frac{\cos^2\frac{\alpha}{2} + \sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}}} = \frac{1 - tg^2\frac{\alpha}{2}}{1 + tg^2\frac{\alpha}{2}}}$$

$$\boxed{\begin{cases} \operatorname{sen}\alpha = \dfrac{2\,tg\frac{\alpha}{2}}{1 + tg^2\frac{\alpha}{2}} = \dfrac{2t}{1+t^2} \\[2ex] \cos\alpha = \dfrac{1 - tg^2\frac{\alpha}{2}}{1 + tg^2\frac{\alpha}{2}} = \dfrac{1-t^2}{1+t^2} \end{cases}}$$

$$\boxed{tg\frac{\alpha}{2} = t}$$

Formule parametriche

$$\boxed{\operatorname{sen}\alpha + \cos\alpha = 0}$$

$$\operatorname{tg}\frac{\alpha}{2} = -\frac{2}{3}$$

$\operatorname{sen}\alpha = ?$

$\cos\alpha = ?$

$$\operatorname{sen}\alpha = \frac{2\operatorname{tg}\frac{\alpha}{2}}{1+\operatorname{tg}^2\frac{\alpha}{2}} = \frac{2\cdot\left(-\frac{2}{3}\right)}{1+\left(-\frac{2}{3}\right)^2} = \frac{-\frac{4}{3}}{1+\frac{4}{9}} = \frac{-\frac{4}{3}}{\frac{9+4}{9}} =$$

$$= -\frac{4}{\cancel{3}_1}\cdot\frac{\cancel{9}^3}{13} = -\frac{12}{13}$$

$$\cos\alpha = \frac{1-\operatorname{tg}^2\frac{\alpha}{2}}{1+\operatorname{tg}^2\frac{\alpha}{2}} = \frac{1-\left(-\frac{2}{3}\right)^2}{1+\left(-\frac{2}{3}\right)^2} = \frac{1-\frac{4}{9}}{1+\frac{4}{9}}$$

$$= \frac{\frac{9-4}{9}}{\frac{9+4}{9}} = \frac{5}{\cancel{9}}\cdot\frac{\cancel{9}}{13} = \frac{5}{13}$$

Formule di bisezione
Bisecare l'angolo significa dividerlo per 2

$$\cos^2\alpha = 1 - \text{sen}^2\alpha$$
$$\text{sen}^2\alpha = 1 - \cos^2\alpha$$

$$\cos 2\alpha = \cos^2\alpha - \text{sen}^2\alpha$$

$$\cos 2\alpha = 1 - \text{sen}^2\alpha - \text{sen}^2\alpha$$
$$\cos 2\alpha = 1 - 2\,\text{sen}^2\alpha$$
$$\boxed{\cos\alpha = 1 - 2\,\text{sen}^2\frac{\alpha}{2}}$$

$$\cos 2\alpha = \cos^2\alpha - (1 - \cos^2\alpha)$$
$$\cos 2\alpha = 2\cos^2\alpha - 1$$
$$\boxed{\cos\alpha = 2\cos^2\frac{\alpha}{2} - 1}$$

DIVIDIAMO α PER 2

$$\cos\alpha = 1 - 2\,\text{sen}^2\frac{\alpha}{2} \implies 2\,\text{sen}^2\frac{\alpha}{2} = 1 - \cos\alpha$$

$$\cos\alpha = 2\cos^2\frac{\alpha}{2} - 1 \implies 2\cos^2\frac{\alpha}{2} = +1 + \cos\alpha$$

$$\begin{cases} \text{sen}^2\frac{\alpha}{2} = \frac{1-\cos\alpha}{2} \implies \text{sen}\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{2}} \\ \cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2} \implies \cos\frac{\alpha}{2} = \pm\sqrt{\frac{1+\cos\alpha}{2}} \end{cases}$$

$$\text{tg}\frac{\alpha}{2} = \frac{\text{sen}\frac{\alpha}{2}}{\cos\frac{\alpha}{2}} = \frac{\cancel{\pm}\sqrt{\frac{1-\cos\alpha}{2}}}{\cancel{\pm}\sqrt{\frac{1+\cos\alpha}{2}}} = \pm\frac{\sqrt{1-\cos\alpha}}{\cancel{\sqrt{2}}}\cdot\frac{\cancel{\sqrt{2}}}{\sqrt{1+\cos\alpha}}$$

$$\boxed{\text{tg}\frac{\alpha}{2} = \pm\sqrt{\frac{1-\cos\alpha}{1+\cos\alpha}}}$$

Formule di bisezione

$\boxed{\operatorname{sen} 22^\circ 30'}$ $\boxed{\cos 22^\circ 30'}$ $\boxed{22^\circ 30' = \frac{45^\circ}{2}}$

$\boxed{\operatorname{tg} 22^\circ 30'}$

$\alpha = 45^\circ$

$$\operatorname{sen} \frac{45^\circ}{2} = + \sqrt{\frac{1-\cos 45^\circ}{2}} = \sqrt{\frac{1-\frac{\sqrt{2}}{2}}{2}} =$$

$$= \sqrt{\frac{\frac{2-\sqrt{2}}{2}}{2}} = \sqrt{\frac{2-\sqrt{2}}{4}} = \frac{\sqrt{2-\sqrt{2}}}{\sqrt{4}}$$

$$= \boxed{\frac{1}{2}\sqrt{2-\sqrt{2}}} \simeq 0,382$$

$$\cos \frac{45^\circ}{2} = \sqrt{\frac{1+\cos 45^\circ}{2}} = \boxed{\frac{1}{2}\sqrt{2+\sqrt{2}} = 0,923}$$

$$\operatorname{tg} \frac{45^\circ}{2} = \sqrt{\frac{1-\cos 45^\circ}{1+\operatorname{sen} 45^\circ}} = \frac{\operatorname{sen}\frac{45^\circ}{2}}{\cos\frac{45^\circ}{2}}$$

$2-\sqrt{}$

$$= \frac{\cancel{\frac{1}{2}}\sqrt{2-\sqrt{2}}}{\cancel{\frac{1}{2}}\sqrt{2+\sqrt{2}}} = \frac{\sqrt{2-\sqrt{2}}}{\sqrt{2+\sqrt{2}}} \cdot \frac{\sqrt{2-\sqrt{2}}}{\sqrt{2-\sqrt{2}}} = \frac{(2-\sqrt{2})}{\sqrt{4-2}}$$