Lezione 14

Preludio ai limiti notevoli

$$\boxed{\log_e e^b = b \log_e e}$$

OSSERVO $\boxed{x = a^{\log_a x}}$ se $\boxed{x > 0}$ $\qquad 2 = \sqrt[2]{(2)^2}$

$\boxed{y = x^\alpha}$ $\qquad \boxed{x \geq 0}$ $\boxed{\alpha > 0}$ $\qquad \alpha \in (\mathbb{R}\setminus\mathbb{Q})^+$ IRRAZ. POSITIVI

$$\boxed{y = x^\alpha = e^{\log_e x^\alpha} = e^{\ln x^\alpha} = e^{\alpha \ln x}}$$

$$\boxed{y = [f(x)]^{g(x)} = e^{\log_e [f(x)]^{g(x)}} = e^{g(x) \log_e f(x)}}$$

ES.

$$\lim_{x \to 1} (2x + \log x)^{x^2+x+1} = (2 + \log 1)^{1+1+1} = 2^3 = \underline{8} \checkmark$$

$$\lim_{x \to 1} e^{(x^2+x+1)\log_e[2x+\log x]} = e^{3 \log_e[2]} = e^{\log_e 2^3} = e^{\log_e 8} = \underline{8} \checkmark$$

LIMITI NOTEVOLI

$\boxed{\begin{array}{l} 1)\quad \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^{x} = e \\ 2)\quad \lim_{z \to 0} \left(1 + z\right)^{\frac{1}{z}} = e \end{array}}$

1^{∞}

1^{∞}

RICORDO

$e \simeq 2{,}718\ldots$

$\frac{1}{x} = z$

$x \to \infty \Rightarrow z \to 0$

1) $\lim_{x \to \infty} \left(1 + \frac{1}{f(x)}\right)^{f(x)} = e$

$\lim_{z \to 0} \left(1 + f(z)\right)^{\frac{1}{f(z)}} = e$

$$\lim_{x\to\infty}\left(\frac{x+1}{x-1}\right)^x =$$

$$\frac{x+1}{x-1} = \frac{x+1-2+2}{x-1} = \frac{x-1+2}{x-1} = \frac{\cancel{x-1}}{\cancel{x-1}} + \frac{2}{x-1} = \boxed{1+\frac{2}{x-1}}$$

$$= 1 + \frac{1}{\frac{x-1}{2}} = 1 + \frac{1}{f(x)} \qquad f(x) = \frac{x-1}{2}$$

$$\lim_{x\to\infty}\left(1+\frac{1}{\frac{x-1}{2}}\right)^x = \lim_{x\to\infty}\left[\left(1+\frac{1}{\frac{x-1}{2}}\right)^{\frac{x-1}{2}}\right]^{x\cdot\frac{2}{x-1}} \to e \quad = \lim_{x\to\infty} e^{\frac{2x}{x-1}} \quad = \underline{e^2}$$

$\to 2$

$$\lim_{x \to -\infty} \left(\frac{2x-3}{x+5}\right)^{x}$$

$$\lim_{x \to \infty} \left(1 + \frac{1}{f(x)}\right)^{f(x)} = 1$$

$$\frac{2x-3}{x+5} = \frac{2x - \cancel{x} + x - 3 + 8 - 8}{x+5} =$$

$$= \frac{\cancel{x+5}}{x+5} + \frac{x-8}{x+5} = 1 + \frac{x-8}{x-5} = \left(1 + \frac{1}{\frac{x-5}{x-8}}\right)^{\frac{x^2-8x}{x-5}} \times \boxed{0}$$

$$\lim_{x \to -\infty} \left(\left(1 + \frac{1}{\frac{x-5}{x-8}}\right)^{\frac{x-5}{x-8}}\right)^{x \cdot \left(\frac{x-8}{x-5}\right)} \Rightarrow e$$

$$= \lim_{x \to -\infty} e^{x \cdot \left(\frac{x-8}{x-5}\right)} = e^{-\infty} = \frac{1}{e^{+\infty}} = \frac{1}{+\infty}$$

$$\boxed{\lim_{x \to 0} \left(1 + x^2\right)^{\frac{1}{x^2}} = e}$$

$$\lim_{x \to 0} \left(1 + f(x)\right)^{\frac{1}{f(x)}}$$

$$f(x) = x^2$$

3 $\boxed{\lim_{x \to 0} \frac{\log_a(1+x)}{x} = \log_a e}$ $\left[\frac{0}{0}\right]$

$$\lim_{x \to 0} \frac{\log_a(1+x)}{x} = \lim_{x \to 0} \frac{1}{x} \cdot \log_a(1+x) = \quad \to e$$

$$= \lim_{x \to 0} \log_a(1+x)^{\frac{1}{x}} = \log_a \boxed{\lim_{x \to 0} (1+x)^{\frac{1}{x}}} = \underline{\log_a e}$$

$a = e$ 4 $\boxed{\lim_{x \to 0} \frac{\log_e(1+x)}{x} = \log_e e = 1}$ $\left[\frac{0}{0}\right]$

5^a $\boxed{\lim_{x \to 0} \frac{a^x - 1}{x} = \log a \quad ; \quad a \in \mathbb{R}^+}$ $\left[\frac{0}{0}\right]$

CAMB. VARIABILE

$a^x - 1 = t \Rightarrow a^x = 1 + t \Longrightarrow x = \log_a(1+t)$

$x \to 0 \Longrightarrow t \to 0$

$\lim_{t \to 0} \frac{t}{\log_a(1+t)} = \frac{1}{\log_a e} = \log_e a$ c.v.d

$\log_2 8 = \frac{1}{\log_8 2}$

6^a $\boxed{\lim_{x \to 0} \frac{e^x - 1}{x} = \log_e e = 1}$ $\left[\frac{0}{0}\right]$

7^a $\boxed{\lim_{x\to 0} \frac{(1+x)^K - 1}{x} = K}$ $\left[\frac{0}{0}\right]$ $K \in \mathbb{R}$

CAMBIAM. DI VARIABILE

$(1+x)^K - 1 = z$

$x \to 0 \Rightarrow z \to 0$ $(1+x)^K = z+1$ PASSIAMO A log

$\log(1+x)^K = \log(z+1)$

$\lim_{\substack{z\to 0\\ x\to 0}} \frac{z}{x} =$ $K \log(1+x) = \log(1+z)$

$= \lim_{\substack{z\to 0\\ x\to 0}} \frac{z}{x} \cdot \frac{K \log(1+x)}{\log(1+z)} = \lim_{z\to 0} \frac{z}{\log(1+z)} \cdot \lim_{x\to 0} \frac{K \log(1+x)}{x}$

$= \frac{1}{\cancel{\log_e e}} \cdot K \cancel{\log_e e} = \boxed{K}$

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$$\lim_{x \to 2} \frac{e^x - e^2}{x-2} =$$

$x - 2 = z$

$x \to 2 \implies z \to 0$

$$\boxed{\lim_{z \to 0} \frac{e^z - 1}{z} = 1}$$

$$= \lim_{z \to 0} \frac{e^{z+2} - e^2}{z} =$$

$$= \lim_{z \to 0} \frac{e^{z+2} - e^2}{z(z+2)} \cdot (z+2) = 1$$

$$= \lim_{z \to 0} \frac{e^{z+2} - e^2}{z+2} \cdot \lim_{z \to 0} \frac{z+2}{2} = \lim_{z \to 0} \frac{\frac{e^{z+2} - 1}{e^2}}{\frac{z+2}{e^2}} =$$

$$\lim_{z \to 0} \frac{e^{z+2} - 1}{z+2} \cdot e^2 = 1 \cdot e^2$$

$$= e^2$$

$$= e^2 \lim_{z \to 0} \frac{\frac{e^{z+2} - \cancel{1}}{e^2}}{\frac{z+2}{e^2}} \to 1 \cdot e^2 = e^2$$

$$\lim_{x \to 0} \frac{3x}{1 - e^{3x}} \cdot \frac{1}{3}$$

$$(-1) \lim_{x \to 0} \frac{3x}{e^{3x} - 1} \cdot \frac{1}{3} = -\frac{1}{3}$$

$\to 1$

$$\boxed{\lim_{x \to 0} \frac{e^{x} - 1}{x} = \lim_{x \to 0} \frac{x}{e^{x} - 1} = 1}$$