Lezione 37

Operazioni sui limiti

$\boxed{\lim f_1(x) = \ell_1}$ $\boxed{\lim f_2(x) = \ell_2}$ $\ell_1 \ell_2 \in \mathbb{R}$

LIMITE DI SOMMA DI FUNZIONI

IP

TS $\lim [f_1(x) \pm f_2(x)] = \ell_1 \pm \ell_2$

Il limite di una somma di funzione è equivalente alla somma dei due limiti

DIM $\boxed{\lim_{x \to c} f_1(x) = \ell_1}$ ε INFINITESIMO

$\forall \frac{\varepsilon}{2} > 0 \; \exists I_1(c) : \forall x \in I_1(c) \Rightarrow |f_1(x) - \ell_1| < \frac{\varepsilon}{2}$

$-\frac{\varepsilon}{2} < f_1(x) - \ell_1 < \frac{\varepsilon}{2}$

$\boxed{\ell_1 - \frac{\varepsilon}{2} < f_1(x) < \ell_1 + \frac{\varepsilon}{2}}$ 1°

$\lim_{x \to c} f_2(x) = \ell_2$

$\forall \frac{\varepsilon}{2} > 0 \; \exists I_2(c) : \forall x \in I_2(c) \Rightarrow |f_2(x) - \ell_2| < \frac{\varepsilon}{2}$

$-\frac{\varepsilon}{2} < f_2(x) - \ell_2 < \frac{\varepsilon}{2}$

$\boxed{\ell_2 - \frac{\varepsilon}{2} < f_2(x) < \frac{\varepsilon}{2}}$ 2°

$I_1(c) \cap I_2(c) = I(c)$

$$+\begin{cases} \ell_1 - \frac{\varepsilon}{2} < f_1(x) < \ell_1 + \frac{\varepsilon}{2} \\ \ell_2 - \frac{\varepsilon}{2} < f_2(x) < \ell_2 + \frac{\varepsilon}{2} \end{cases}$$

$$\left(\ell_1 - \frac{\varepsilon}{2}\right) + \left(\ell_2 - \frac{\varepsilon}{2}\right) < f_1(x) + f_2(x) < \left(\ell_1 + \frac{\varepsilon}{2}\right) + \left(\ell_2 - \frac{\varepsilon}{2}\right)$$

$$\boxed{(\ell_1 + \ell_2) - \varepsilon < f_1(x) + f_2(x) < \ell_1 + \ell_2 + \varepsilon}$$

$$\lim_{x \to c} f_1(x) + f_2(x) = \ell_1 + \ell_2$$

$$\lim_{x \to 4} \Big(\boxed{\underbrace{\sqrt{x}}_{f_1(x)} + \underbrace{\log_4 x}_{f_2(x)}} \Big) = \lim_{x \to 4} \sqrt{x} + \lim_{x \to 4} \log_4 x$$

$$= \underbrace{\sqrt{4}} + \log_4 4 = 2 + 1 = \underline{3}$$

$$\log_2 8 = 3 \qquad 2^3 = 8$$

$$4^{x} = 4$$

$$\lim_{x \to \pi} (\underbrace{x}_{f_1(x)} + \underbrace{5}_{f_2(x)}) = \lim_{x \to \pi} x + \lim_{x \to \pi} 5 = \underline{\pi + 5}$$

Forme indeterminate nel caso della somma algebrica fra limiti

1) $\lim f_1(x) = l \quad ; \quad \lim f_2(x) = \underline{\pm \infty}$

$$\lim \left[f_1(x) + f_2(x) \right] = l + (\pm \infty) = \underline{\pm \infty}$$

$$\lim \left[f_1(x) - f_2(x) \right] = l - (\pm \infty) = \underline{\mp \infty}$$

2) $\underline{\lim f_1(x) = +\infty} \qquad \underline{\lim f_2(x) = +\infty}$

$$\lim \left[f_1(x) + f_2(x) \right] = \underline{+\infty} + \underline{(+\infty)} = \underline{+\infty}$$

$$\lim \left[f_1(x) - f_2(x) \right] = +\infty + (-\infty) = \boxed{+\infty - \infty}$$

FORMA INDETERM.

3) $\lim f_1(x) = -\infty \qquad \lim f_2(x) = -\infty$

$$\lim \left[f_1(x) + f_2(x) \right] = -\infty + (-\infty) = -\infty - \infty = \underline{-\infty}$$

$$\lim \left[f_1(x) - f_2(x) \right] = -\infty - (-\infty) = \boxed{-\infty + \infty}$$

FORMA INDETERM.

4) $\lim f_1(x) = +\infty \qquad \lim f_2(x) = -\infty$

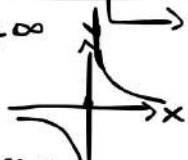
$$\lim \left[f_1(x) + f_2(x) \right] = +\infty + (-\infty) = \boxed{+\infty - \infty}$$

FORMA INDETERM.

$$\lim \left[f_1(x) - f_2(x) \right] = +\infty - (-\infty) = +\infty + \infty = +\infty$$

$$\lim_{x \to 0^+} \left(\underset{f_1(x)}{\frac{1}{x}} + \underset{f_2(x)}{\operatorname{sen} x} \right) = \underbrace{\lim_{x \to 0^+} \frac{1}{x}}_{} + \underbrace{\lim_{x \to 0} \operatorname{sen} x}_{\to 0}$$

$$= +\infty + 0 = +\infty$$



$$\lim_{x \to +\infty} (x + \operatorname{sen} x) = \underbrace{\lim_{x \to +\infty} x}_{+\infty} + \underbrace{\lim_{x \to +\infty} \operatorname{sen} x}_{(+\infty)} =$$

$$+\infty + (+\infty) = +\infty$$

Limite di un prodotto di funzioni

Ip $\lim f_1(x) = l_1$; $\lim f_2(x) = l_2$

Ts $\lim [f_1(x) \cdot f_2(x)] = l_1 \cdot l_2$

Dim $\lim [f_1(x) - l_1] = 0$ $\quad \lim [f_2(x) - l_2] = 0$

$$\varphi(x) = (f_1(x) - l_1)(f_2(x) - l_2) =$$

$$= f_1(x) f_2(x) - l_2 f_1(x) - l_1 f_2(x) + l_1 l_2$$

$$f_1(x) f_2(x) = \varphi(x) + l_2 f_1(x) + l_1 f_2(x) - l_1 l_2$$

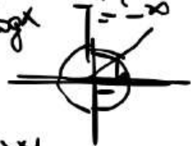
$$\lim f_1(x) f_2(x) = \underbrace{\lim \varphi(x)}_{0} + \underbrace{l_2 \underbrace{\lim f_1(x)}_{l_1}}_{l_2 l_1} + \underbrace{l_1 \underbrace{\lim f_2(x)}_{l_2}}_{l_1 l_2} - l_1 l_2$$

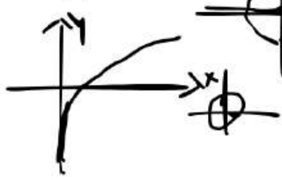
$$= 0 + l_1 l_2 + \cancel{l_1 l_2} - \cancel{l_1 l_2} = l_1 l_2$$

c.v.d.

$$\lim_{x\to 0} \underbrace{x}_{f_1(x)}\, \underbrace{e^x}_{f_2(x)} = \lim_{x\to 0} x \cdot \lim_{x\to 0} e^x = 0 \cdot \underbrace{e^0}_{} = 0 \cdot 1 = 0$$

$$\lim_{x\to 0} \underbrace{\cos x}_{f_1(x)}\, \underbrace{\log x}_{f_2(x)} = \lim_{x\to 0} \overset{\to 1}{\cos x} \cdot \lim_{x\to 0} \overset{\to -\infty}{\log x} = 1\cdot(-\infty) = -\infty$$





Forme indeterminate del prodotto

$\boxed{\infty \cdot \infty = \infty}$

FORMA INDETERM

$\lim f_1(x) = \infty$
$\lim f_2(x) = \infty$

LIMITE DEL QUOZIENTE DI DUE FUNZIONI

Ip $\boxed{\lim_{x \to c} f(x) = \ell}$ $\quad (\ell \neq 0)$

Ts $\lim_{x \to c} \frac{1}{f(x)} = \frac{1}{\ell}$

Dim

$\forall\, \varepsilon > 0 \ \exists\, I(c) : \forall x \in I(c) \Rightarrow |f(x) - \ell| < \varepsilon$

$\boxed{|f(x)| > \frac{|\ell|}{2}}$ $\quad f(x) > \frac{\ell}{2} = \frac{|\ell|}{2}$ $\quad f(x) < \frac{\ell}{2} = -\frac{|\ell|}{2}$

DALLA PERMANENZA DEL SEGNO

$\left|\frac{1}{f(x)} - \frac{1}{\ell}\right| = \left|\frac{\ell - f(x)}{\ell f(x)}\right| = \frac{|\ell - f(x)|}{|\ell| \cdot |f(x)|}$

$\frac{|\ell - f(x)|}{|\ell|\,|f(x)|} < \frac{\varepsilon}{|\ell| \cdot \frac{|\ell|}{2}} = \frac{2\varepsilon}{\ell^2} \Rightarrow \left|\frac{1}{f(x)} - \frac{1}{\ell}\right| < \frac{2\varepsilon}{\ell^2}$

$\forall\, \frac{2\varepsilon}{\ell^2} \ \exists\, I(c) \ \forall x \in I(c) \Rightarrow \left|\frac{1}{f(x)} - \frac{1}{\ell}\right| < \frac{2\varepsilon}{\ell}$

$\boxed{\lim_{x \to c} \frac{1}{f(x)} = \frac{1}{\ell}}$ c.v.d

$\boxed{\lim \frac{f_1(x)}{f_2(x)} = \frac{\lim f_1(x)}{\lim f_2(x)}}$

Forme indeterminate nel quoziente

1) $\lim f_1(x) = \ell \neq 0$ $\quad \lim f_2(x) = 0 \Rightarrow \lim \frac{f_1(x)}{f_2(x)} \frac{\ell}{0} = \infty$

2) $\lim f_1(x) = \infty$ $\quad \lim f_2(x) = \ell \Rightarrow \lim \frac{f_1(x)}{f_2(x)} = \frac{\infty}{\ell} = \infty$

$\frac{\infty}{0} = \frac{\infty}{\ell} = \infty$

3) $\lim f_1(x) = \ell$ $\quad (\ell = 0, \ \ell \neq 0)$ $\quad \lim f_2(x) = \infty \Rightarrow \lim \frac{f_1(x)}{f_2(x)} = \frac{\ell}{\infty}$

$\frac{0}{\infty} = \frac{\ell}{\infty} = 0 \quad = 0$

4) $\lim f_1(x) = 0$ $\quad \lim f_2(x) = 0 \Rightarrow \lim \frac{f_1(x)}{f_2(x)} \left[\frac{0}{0}\right]$

FORMA INDETERMINATA

$\left[\frac{\infty}{\infty}\right]$

$\lim_{x \to 1} \frac{e^x}{x} = \frac{\lim_{x \to 1} e^x}{\lim_{x \to 1} x} = \frac{e}{1} = e$

$e^{-x} = \frac{1}{e^x}$

$\lim_{x \to +\infty} \frac{\log x}{e^{-x}} = \frac{+\infty}{0} = +\infty$ $\quad \frac{1}{+\infty}$

$\lim_{x \to 2} \frac{x}{x-2} = \frac{2}{2-2} = \frac{2}{0} = \infty$