Lezione 19

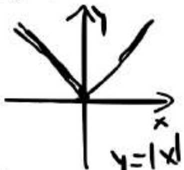
Esempi di non derivabilità delle funzioni continue

$$|x| = \begin{cases} x & \text{se } x \geq 0 \\ -x & \text{se } x < 0 \end{cases}$$

$$y = f(x) = |x|$$

$$\exists \lim_{x \to 0} |x| = 0 = \underline{f(0)} \implies f \text{ CONTINUA IN } x = 0$$

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{|h|}{h}$$



$y = |x|$

$$f'_+(0) \lim_{h \to 0^+} \frac{\cancel{h}}{\cancel{h}} = 1 \qquad f'_-(0) \lim_{h \to 0^-} \frac{-\cancel{h}}{\cancel{h}} = -1$$

$f'_+(0) = 1$

$f'_-(0) = -1$

$y = f(x) = \sqrt[3]{x}$ $\qquad$ Dom $f = \mathbb{R}$

$\exists! \lim_{x \to 0} \sqrt[3]{x} = 0 = f(0)$ $\qquad$ f CONTINUA IN $x = 0$

$$f'(0) = \lim_{h \to 0} \frac{f(0+h) - f(0)}{h} = \lim_{h \to 0} \frac{\sqrt[3]{h}}{h} =$$

$$= \lim_{h \to 0} \frac{h^{\frac{1}{3}}}{h} = \lim_{h \to 0} \frac{1}{h^{\frac{2}{3}}} = \frac{1}{0} \to +\infty$$

$\frac{1}{3} - 1 = \frac{1-3}{3} = -\frac{2}{3}$

$\nexists \; f'(0)$

Derivate fondamentali

Derivata funzione costante

$c \in \mathbb{R}$

$y = f(x) = c$

$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{c - c}{h} =$

$= \lim_{h \to 0} \frac{0}{h} = 0$

$\boxed{f'(x) = 0 \quad f(x) = c}$

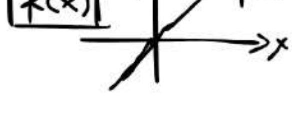
Derivata di una funzione lineare

$$y = f(x) = x$$

$$f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{\cancel{x}+h-\cancel{x}}{h} =$$

$$= \lim_{h\to 0} \frac{\cancel{h}^{\,1}}{\cancel{h}_{\,1}} = 1$$

$$\boxed{f'(x)=1 \quad f(x)=x}$$



Derivata della funzione potenza con esponente intero e positivo

$n \in \mathbb{N}$

$$y = f(x) = x^n$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

1
1 2 1
1 3 3 1
1 4 6 4 1

$$\boxed{(x+h)^n = x^n + \underbrace{n_1 x^{n-1} h + n_2 x^{n-2} h^2 + \dots + n_{n-1} x h^{n-1} + h^n}}$$

Regola generale sviluppo potenza binomiale

$n_1, n_2, \dots n_{n-1} \in \mathbb{N}$

$$(x+h)^n - x^n = n_1 x^{n-1} h + n_2 x^{n-2} h^2 + \dots + n_{n-1} x h^{n-1} + h^n$$

$$\frac{n_1 x^{n-1} h + n_2 x^{n-1} h^2 + \dots - n_{n-1} x h^{n-1} + h^n}{h} =$$

$$= n_1 x^{n-1} + n_2 x^{n-1} \cdot h + \dots \; n_{n-1} x h^{n-2} + h^{n-1}$$

$$= n x^{n-1} + n_2 x^{n-1} h + \dots + n_{n-1} x h^{n-2} + h^{n-1} \qquad \underline{n_1 = n}$$

per $h \to 0$

$$\boxed{f'(x) = n x^{n-1} \qquad f(x) = x^n}$$

$\underline{n=2}$ $\quad f(x) = x^2 \qquad f'(x) = 2x^{2-1} = 2x$

$\underline{n=3}$ $\quad f(x) = x^3 \qquad f'(x) = 3x^{3-1} = 3x^2$

$\underline{n=4}$ $\quad f(x) = x^4 \qquad f'(x) = 4x^{4-1} = 4x^3$

Derivata della funzione radice quadrata di x

$$\mathrm{Dom}\, f = \mathbb{R}^+$$

$$y = \sqrt{x}$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$\lim_{h \to 0} \frac{(\sqrt{x+h} - \sqrt{x})}{h} \frac{(\sqrt{x+h} + \sqrt{x})}{(\sqrt{x+h} + \sqrt{x})} =$$

$$= \lim_{h \to 0} \frac{\cancel{x} + h - \cancel{x}}{h(\sqrt{x+h} + \sqrt{x})} = \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h} + \sqrt{x})} =$$

$$= \lim_{h \to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} = \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

$$\boxed{f'(x) = \frac{1}{2\sqrt{x}} \qquad f(x) = \sqrt{x}}$$

Derivata funzione seno

$$y = f(x) = \operatorname{sen} x$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\operatorname{sen}(x+h) - \operatorname{sen} x}{h}$$

PROSTAFERESI

$$\boxed{\operatorname{sen} p - \operatorname{sen} q = 2 \operatorname{sen} \frac{p-q}{2} \cos \frac{p+q}{2}}$$

$$\operatorname{sen}(x+h) - \operatorname{sen} x = 2 \operatorname{sen} \left(\frac{\cancel{x} + h - \cancel{x}}{2}\right) \cos \left(\frac{x+h+x}{2}\right)$$

$$= \boxed{2 \operatorname{sen} \frac{h}{2} \cos \left(\frac{2x+h}{2}\right)}$$

$\cos x$

$$\lim_{h \to 0} \frac{2 \operatorname{sen} \frac{h}{2} \cos \left(\frac{2x+h}{2}\right)}{h} = \left(\lim_{h \to 0} \frac{2 \operatorname{sen} \frac{h}{2}}{h}\right) \lim_{h \to 0} \cos \left(\frac{2x+h}{2}\right)$$

$$\cos \left(\frac{2x}{2}\right)$$

$$= \cos x \lim_{h \to 0} \frac{\operatorname{sen} \frac{h}{2}}{\frac{h}{2}} \to 1 = \cos x$$

$$\lim_{x \to 0} \frac{\operatorname{sen} x}{x} = 1$$

$$\boxed{f(x) = \operatorname{sen} x \quad ; \quad f'(x) = \cos x}$$

Derivata della funzione coseno

$$y = \cos x$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} =$$

$$\cos p - \cos q = -2\,\mathrm{sen}\frac{p-q}{2}\,\mathrm{sen}\frac{p+q}{2}$$

$$= \lim_{h \to 0} \frac{\cos(x+h) - \cos x}{h}$$

$$= \lim_{h \to 0} \frac{-2\,\mathrm{sen}\left(\frac{\cancel{x}+h-\cancel{x}}{2}\right)\mathrm{sen}\left(\frac{x+h+x}{2}\right)}{h} =$$

$$= -\lim_{h \to 0} \frac{2\,\mathrm{sen}\frac{h}{2}\,\mathrm{sen}\left(\frac{2x+h}{2}\right)}{h} =$$

$$= -\lim_{h \to 0} \left(\frac{\mathrm{sen}\frac{h}{2}}{\frac{h}{2}}\right) \to 1 \quad \cdot \quad \lim_{h \to 0} \mathrm{sen}\left(\frac{2x+h}{2}\right) \to \mathrm{sen}\frac{\cancel{2}x}{\cancel{2}} \to \mathrm{sen}\,x$$

$$= -\mathrm{sen}\,x$$

$$\boxed{f'(x) = -\mathrm{sen}\,x \qquad f(x) = \cos x}$$

Derivata della funzione logaritmica

$$y = f(x) = \log_a x$$

$a \in \mathbb{R}^+ \setminus \{0, 1\}$

$\text{Dom } f = \mathbb{R}^+ \setminus \{0\}$

$$\boxed{\log b - \log a = \log \frac{b}{a}}$$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} =$$

$$\lim_{h \to 0} \frac{\log_a (x+h) - \log_a x}{h}$$

$$\lim_{h \to 0} \frac{1}{h} \log_a \left(\frac{x+h}{x}\right) = \lim_{h \to 0} \frac{1}{h} \log_a \left(1 + \frac{h}{x}\right) =$$

$$= \lim_{h \to 0} \frac{1}{x} \cdot \frac{x}{h} \log_a \left(1 + \frac{h}{x}\right) =$$

$$\boxed{b \log_a c = \log_a c^b}$$

$$= \frac{1}{x} \lim_{h \to 0} \frac{x}{h} \log_a \left(1 + \frac{h}{x}\right) =$$

$\frac{x}{h} = z$

$h \to 0 \Rightarrow z \to \infty$

$$= \frac{1}{x} \lim_{h \to 0} \log_a \left(1 + \frac{h}{x}\right)^{\frac{x}{h}}$$

$$= \frac{1}{x} \lim_{z \to \infty} \log_a \left(1 + \frac{1}{z}\right)^z = \to e$$

PER LA CONTINUITÀ DI $f(g(x))$

$$= \frac{1}{x} \log_a \left[\lim_{z \to \infty} \left(1 + \frac{1}{z}\right)^z\right]$$

$$= \boxed{\frac{1}{x} \log_a e}$$

$$y = \log_a x$$

$$f'(x) = \frac{1}{x} \log_a e$$

se $a = e$ $\quad f'(x) = \frac{1}{x} \log_e e = \frac{1}{x}$

$\boxed{\ln = \log_e}$ $\quad \boxed{y = \ln x \quad f'(x) = \frac{1}{x}}$

Derivata funzione esponenziale

$y = f(x) = a^x \qquad \text{Dom} f = \mathbb{R} \qquad a \in \mathbb{R}^+$

$$\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} =$$

$$= \lim_{h \to 0} \frac{a^{x+h} - a^x}{h} = \lim_{h \to 0} \frac{a^x \cdot a^h - a^x}{h} =$$

$$= \lim_{h \to 0} \frac{a^x (a^h - 1)}{h} = a^x \cdot \boxed{\lim_{h \to 0} \frac{a^h - 1}{h}} \to \log_e a$$

$$\boxed{f(x) = a^x \qquad f'(x) = a^x \log_e a}$$

se $a = e$

$f(x) = e^x \qquad f'(x) = e^x \log_e e \to 1$

$$\boxed{f(x) = e^x \qquad f'(x) = e^x}$$

$$\boxed{f(x) = 2^x \qquad f'(x) = 2^x \ln 2}$$