Lezione 29

Integrazione funzioni razionali fratte

$\boxed{\text{grado } N(x) > \text{grado } D(x)}$

$$y = \frac{N(x)}{D(x)} =$$

$$y = \frac{Q(x) \cdot D(x) + R(x)}{D(x)} = Q(x) + \frac{R(x)}{D(x)}$$

$$f(x) = Q(x) + \frac{R(x)}{D(x)}$$

$$\int f(x)\,dx = \int Q(x)\,dx + \int \frac{R(x)}{D(x)}\,dx$$

$N(x) = 4x^4 + 8x^2 + x + 3$

$D(x) = 2x^2 + 1$

ES. $$\int \frac{4x^4 + 8x^2 + x + 3}{2x^2+1}\,dx$$

$$\begin{array}{rrrrr|l} 4x^4 & +0x^3 & +8x^2 & +x & +3 & 2x^2+1 \\ \hline -4x^4 & & -2x^2 & & & 2x^2+3 \\ \hline = & = & 6x^2 & +x & +3 & \\ & & -6x^2 & & -3 & \\ \hline & & // & x & // & \end{array}$$

$Q(x) = 2x^2 + 3$

$R(x) = x$

$D(x) = 2x^2 + 1$

$N(x) = Q(x)\,D(x) + R(x)$

$$\int \frac{(2x^2+3)(2x^2+1)+x}{2x^2+1}\,dx =$$

$$= \int \frac{(2x^2+3)\cancel{(2x^2+1)}}{\cancel{(2x^2+1)}}\,dx + \frac{1}{2}\int \frac{2x}{2x^2+1}\,dx =$$

$$= \int (2x^2+3)\,dx + \frac{1}{2}\int \frac{2x}{2x^2+1}\,dx$$

$$\int \frac{f'(x)\,dx}{f(x)} = \log|f(x)| + c$$

$$= 2\int x^2\,dx + 3\int dx + \frac{1}{2}\log|f(x)| + c$$

$$= 2 \cdot \left[\frac{x^{2+1}}{2+1} + c\right] + 3x + c + \frac{1}{2}\log|2x^2+1| + c$$

$$= \boxed{\frac{2}{3}x^3 + 3x + \frac{1}{2}\log|2x^2+1| + c}$$

$\text{grado}[N(x)] < \text{grado}[D(x)]$

(A) $\int \frac{px+q}{ax^2+bx+c}\,dx$ (B) $\int \frac{q}{ax^2+bx+c}\,dx$

$a \in \mathbb{R}$

1ª CASO $\Delta > 0$

$$\boxed{\begin{aligned} &a(x-x_1)(x-x_2) \\ &= ax^2+bx+c \end{aligned}}$$

$$\frac{1}{ax^2+bx+c} = \frac{1}{a(x-x_1)(x-x_2)} = \frac{A}{x-x_1} + \frac{B}{x-x_2}$$

$$\frac{A(x-x_2)+B(x-x_1)}{(x-x_1)(x-x_2)}$$

ID. POLINOMIALE

$$\int \frac{2x-7}{x^2-x-2}\,dx$$

$$= \int \frac{2x-7}{(x+1)(x-2)}\,dx$$

$$\Delta = (-1)^2 - 4(1)(-2) = 1 + 8 = 9 > 0$$

$$x_1, x_2 = \frac{1 \pm 3}{2} \begin{cases} \frac{\cancel{2}}{\cancel{2}} = -1 \\ \frac{\cancel{4}^2}{\cancel{2}_1} = 2 \end{cases}$$

$x = -1$ $\quad x = 2$

$x + 1 = 0$ $\quad x - 2 = 0$

$(x+1)(x-2) =$

$= x^2 - x - 2$

$$\frac{2x-7}{(x+1)(x-2)} = \frac{A}{(x+1)} + \frac{B}{(x-2)} = \frac{A(x-2)+B(x+1)}{(x+1)(x-2)}$$

$$\frac{2x-7}{(x+1)(x-2)} = \frac{(A+B)x - 2A + B}{(x+1)(x-2)}$$

$$\begin{cases} A+B = 2 \\ -2A+B = -7 \end{cases}$$

$R_1 - R_2$ $\quad 3A = 9 \Rightarrow A = \frac{\cancel{9}^3}{\cancel{3}_1}$

$A = 3$

$B = 2 - A = 2 - 3 = -1$

$B = -1$

$$\frac{2x-7}{x^2-x-2} = \frac{3}{x+1} - \frac{1}{x-2}$$

$$\int \frac{f'(x)}{f(x)}\,dx$$

$$\int \frac{2x-7}{x^2-x-2}\,dx = \int \frac{3}{x+1}\,dx - \int \frac{1}{x-2}\,dx$$

$$= 3\int \frac{1}{x+1}\,dx - \log|x-2| + C$$

$$= 3\log|x+1| - \log|x-2| + C$$

$$= \log\frac{|x+1|^3}{|x-2|} + C$$

2ª CASO $\Delta = 0$ $\boxed{x_1 = x_2}$

$$ax^2+bx+c = a(x-x_1)^2$$

(B)

$$\int \frac{q}{ax^2+bx+c}\,dx = \int \frac{q}{a(x-x_1)^2}\,dx = \frac{q}{a}\int (x-x_1)^{-2}dx$$

$$= \frac{q}{a}\,\frac{(x-x_1)^{-2+1}}{-2+1} + C = \frac{q}{a}\,\frac{(x-x_1)^{-1}}{-1} + C = \boxed{-\frac{q}{a(x-x_1)} + C}$$

$$\int \frac{px+q}{ax^2+bx+c}\,dx = \int \frac{px+q}{a(x-x_1)^2}\,dx$$

$$\frac{px+q}{a(x-x_1)^2} = \frac{A}{a(x-x_1)} + \frac{B}{a(x-x_1)^2} = \frac{A(x-x_1)+B}{a(x-x_1)^2}$$

$$\begin{cases} A = p \\ -Ax_1 + B = q \end{cases}$$

ES.

$$\int \frac{dx}{4x^2-4x+1} = \int \frac{1}{4x^2-4x+1}\,dx =$$

$$(4x^2-4x+1) = (2x-1)^2$$

$$= \int \frac{1}{(2x-1)^2}\,dx = \frac{1}{2}\int (2x-1)^{-2}\cdot 2\,dx$$

$$= \frac{1}{2}\cdot\frac{(2x-1)^{-1}}{-1} + C = -\frac{1}{2}\cdot\frac{1}{(2x-1)} + C$$

$$= -\frac{1}{2(2x-1)} + C$$

$$\boxed{\int [f(x)]^{\alpha} f'(x)\,dx = \frac{[f(x)]^{\alpha+1}}{\alpha+1} + C}$$

$$\int \frac{x+5}{x^2-6x+9}\,dx = \int \frac{x+5}{(x-3)^2}\,dx \qquad \Delta = 0$$

$$\frac{x+5}{(x-3)^2} = \frac{A}{x-3} + \frac{B}{(x-3)^2} = \frac{A(x-3)+B}{(x-3)^2}$$

$$\frac{1x+5}{(x-3)^2} = \frac{Ax-3A+B}{(x-3)^2}$$

$$\begin{cases} A=1 \\ -3A+B=5 \end{cases} \Longrightarrow -3+B=5 \Longrightarrow B=8$$

$$\begin{cases} A=1 \\ B=8 \end{cases}$$

$$\frac{x+5}{(x-3)^2} = \frac{1}{x-3} + \frac{8}{(x-3)^2}$$

$$\int \frac{x+5}{(x-3)^2}\,dx = \int \frac{1}{x-3}\,dx + 8\int \frac{1}{(x-3)^2}\,dx$$

$$= \log|x-3| + 8\int (x-3)^{-2}\,dx + c$$

$$= \log|x-3| + 8\,\frac{(x-3)^{-1}}{-1} + c$$

$$= \boxed{\log|x-3| - \frac{8}{x-3} + c}$$

3° caso $\Delta < 0$

$$ax^2 + bx + c > 0$$

$$a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right) > 0$$

$$a\left[\underbrace{(x)^2}_{\text{quadrato}} + \underbrace{2\cdot\frac{b}{2a}x}_{\text{doppio prodotto}} + \underbrace{\frac{b^2}{4a^2}}_{\text{quadrato}} - \frac{b^2}{4a^2} + \frac{c}{a}\right] > 0$$

$$a\left[\left(x + \frac{b}{2a}\right)^2 + \frac{-b^2 + 4ac}{4a^2}\right] > 0 \qquad -\Delta$$

$$a\left[\underbrace{\left(x + \frac{b}{2a}\right)^2}_{>0} \underbrace{- \frac{\Delta}{4a^2}}_{>0}\right] > 0 \qquad \Delta < 0$$

$$ax^2 + bx + c = a\left[\left(x + \frac{b}{2a}\right)^2 - \frac{\Delta}{4a^2}\right]$$

$$\begin{cases} \dfrac{b}{2a} = K \\ \dfrac{\sqrt{-\Delta}}{2a} = m \end{cases} \qquad \boxed{\Delta = b^2 - 4ac}$$

$$\boxed{ax^2 + bx + c = a\left[(x+K)^2 + m^2\right]}$$ decomp. non scomp.

(B) $$\int \frac{q}{ax^2 + bx + c}\,dx = \int \frac{q}{ax^2 + bx + c}\,dx = \quad q = 1$$

$$= \int \frac{q}{a\left[(x+K)^2 + m^2\right]}\,dx = \int \frac{1}{(x+K)^2 + m^2}\,dx$$

$$\int \frac{1}{m^2\left[1 + \left(\frac{x+K}{m}\right)^2\right]}\,dx = \qquad \int \frac{f'(x)}{m^2\left[1 + f(x)^2\right]}\,dx = \operatorname{arctg} f(x) + c$$

$$= \frac{1}{m}\int \frac{\frac{1}{m}}{\left[1 + \left(\frac{x+K}{m}\right)^2\right]}\,dx = \frac{1}{m}\operatorname{arctg}\left(\frac{x+K}{m}\right) + C$$

(A) $$\int \frac{px + q}{ax^2 + bx + c}\,dx \qquad \Delta < 0$$

$$\frac{px + q}{ax^2 + bx + c} = \frac{p\left(x + \frac{q}{p}\right)}{ax^2 + bx + c} = \frac{\frac{p}{2a}\left(2ax + \frac{2aq}{p}\right)}{ax^2 + bx + c} =$$

$$= \frac{\frac{p}{2a}\left[2ax + b + \frac{2aq}{p} - b\right]}{ax^2 + bx + c} =$$

$$= \frac{p}{2a}\left[\frac{2ax + b}{ax^2 + bx + c} + \frac{\frac{2aq}{p} - b}{ax^2 + bx + c}\right]$$

$$\int \frac{3x+5}{4x^2+12x+11}\,dx \qquad \Delta<0 \qquad a)$$

$p=3$
$q=5$
$a=4$
$b=12$

$$\frac{3x+5}{4x^2+12x+11} = \frac{p}{2a}\left[\frac{2ax+b}{ax^2+bx+c} + \frac{\frac{2aq}{p}-b}{ax^2+bx+c}\right] =$$

$$= \frac{3}{8}\left[\frac{8x+12}{4x^2+12x+11} + \frac{\frac{40}{3}-12}{4x^2+12x+11}\right]$$

$$= \frac{3}{\cancel{8}_2}\cdot\frac{\cancel{4}(2x+3)}{4x^2+12x+11} + \frac{3}{8}\cdot\frac{\frac{40-36}{3}}{4x^2+12x+11} =$$

$$= \frac{3}{2}\,\frac{(2x+3)}{4x^2+12x+11} + \frac{\cancel{3}}{\cancel{8}_2}\cdot\frac{\frac{\cancel{4}^1}{\cancel{3}}}{4x^2+12x+11} =$$

$$\frac{3}{4}\int\frac{4\cdot 2x+3}{\underbrace{4x^2+12x+11}}\,dx + \frac{1}{2}\int\frac{1}{4x^2+12x+11}\,dx =$$

$$= \frac{3}{8}\int\frac{8x+12}{4x^2+12x+11}\,dx + \frac{1}{2}\int\frac{1}{4\left(x^2+3x+\frac{11}{4}\right)}\,dx$$

$$\boxed{\int\frac{f'(x)}{f(x)}\,dx}$$

$$= \log|f(x)|$$

$$\frac{3}{8}\log|4x^2+12x+11|$$

$$\frac{1}{2}\cdot\frac{1}{4}\int\frac{1}{x^2+3x+\frac{11}{4}}\,dx$$

$$2\cdot\frac{3}{2}x$$

$$\frac{1}{8}\int\frac{1}{x^2+3x+\frac{9}{4}-\frac{9}{4}+\frac{11}{4}}\,dx$$

$$+\frac{\sqrt{2}}{8}\,\mathrm{arctg}\,\frac{2x+3}{\sqrt{2}}+C$$