Lezione 31

$$\int \frac{1}{\sqrt{3-x^2}}\,dx = \int \frac{1}{\sqrt{3\left(1-\frac{x^2}{3}\right)}}\,dx = \frac{1}{\sqrt{3}}\int \frac{1}{\sqrt{1-\left(\frac{x}{\sqrt{3}}\right)^2}}\,dx =$$

$$= \frac{1}{\cancel{\sqrt{3}}}\,\cancel{\sqrt{3}}\int \frac{\frac{1}{\sqrt{3}}}{\sqrt{1-\left(\frac{x}{\sqrt{3}}\right)^2}}\,dx = \arcsen\left(\frac{x}{\sqrt{3}}\right) + c \qquad t = \frac{x}{\sqrt{3}}$$

$$\int \frac{f'(x)}{\sqrt{1-[f(x)]^2}}\,dx = \arcsen f(x) + c$$

$$x = \sqrt{3}\,t \implies dx = \sqrt{3}\,dt$$

$$\int \frac{1}{\sqrt{3-x^2}}\,dx = \int \frac{1}{\sqrt{3-3t^2}}\sqrt{3}\,dt = \sqrt{3}\int \frac{1}{\sqrt{3}\sqrt{1-t^2}}\,dt = \frac{\cancel{\sqrt{3}}}{\cancel{\sqrt{3}}}\int \frac{1}{\sqrt{1-t^2}}\,dt$$

$$= \arcsen t + c = \arcsen\left(\frac{x}{\sqrt{3}}\right) + c$$

$$\boxed{\int \sqrt{a^2-x^2}\,dx = \frac{1}{2}a^2 \arcsen\frac{x}{a} + \frac{1}{2}\sqrt{a^2-x^2} + C \qquad a>0}$$

Dim

$$\underline{x = a\,\mathrm{sen}\,t \implies dx = a\cos t\,dt} \qquad t = \arcsen\frac{x}{a}$$

$$\int \sqrt{a^2 - a^2\,\mathrm{sen}^2 t}\; a\cos t\,dt = \int \sqrt{a^2(1-\mathrm{sen}^2 t)}\cdot a\cdot \cos t\,dt =$$

$$= a\cdot a\int \sqrt{1-\mathrm{sen}^2 t}\cdot a\cdot \cos t\,dt =$$

$$= a^2\int \cos t\,\cos t\,dt = a^2\int \cos^2 t\,dt =$$

$$= a^2\int \frac{1+\cos 2t}{2}\,dt =$$

$$= a^2\int \frac{1}{2}\,dt + a^2\,\frac{1}{2}\int \cos 2t\,dt =$$

$$= \frac{1}{2}a^2\cdot t + \frac{1}{2}a^2\,\frac{1}{2}\int \cos 2t\; 2\,dt$$

$$= \frac{1}{2}a^2 t + \frac{1}{4}a^2\,\mathrm{sen}\,2t + c$$

$$= \frac{1}{2}a^2 \arcsen\frac{x}{a} + \frac{1}{4}a^2\,\mathrm{sen}\boxed{2\left(\arcsen\frac{x}{a}\right)} + c$$

$$= \frac{1}{2}a^2 \arcsen\frac{x}{a} + \frac{1}{\cancel{4}_2}a^2\,\cancel{2}\,\mathrm{sen}\,\theta\cos\theta + c$$

$$= \frac{1}{2}a^2 \arcsen\frac{x}{a} + \frac{1}{2}a^2\,\mathrm{sen}\,\theta\sqrt{1-\mathrm{sen}^2\theta} + c =$$

$$= \frac{1}{2}a^2 \arcsen\frac{x}{a} + \frac{1}{2}a^2\,\mathrm{sen}\left(\arcsen\frac{x}{a}\right)\cdot\sqrt{1-\mathrm{sen}^2\left(\arcsen\frac{x}{a}\right)} + c$$

$$= \frac{1}{2}a^2 \arcsen\frac{x}{a} + \frac{1}{2}\cancel{a^2}\cdot\frac{x}{\cancel{a}}\cdot\sqrt{1-\frac{x^2}{a^2}} + c =$$

$$= \frac{1}{2}a^2 \arcsen\frac{x}{a} + \frac{1}{2}a x\cdot\sqrt{\frac{a^2-x^2}{a^2}} + c =$$

$$= \frac{1}{2}a^2 \arcsen\frac{x}{a} + \frac{1}{2}\frac{\cancel{a}}{\cancel{a}}x\sqrt{a^2-x^2} + c =$$

$$= \frac{1}{2}a^2 \arcsen\frac{x}{a} + \frac{1}{2}x\sqrt{a^2-x^2} + c$$

$$\boxed{\begin{aligned}&\cos 2t = 2\cos^2 t - 1\\ &\text{DUPLIC. DEL COSENO}\\ &\cos^2 t = \frac{\cos 2t + 1}{2}\\ &\cos^2 t = \frac{1+\cos 2t}{2}\end{aligned}}$$

$$\int \cos f(t)\cdot f'(t)\,dt = \mathrm{sen}\,f(t) + c$$

$$\boxed{\begin{aligned}&\mathrm{sen}\,2\theta\\ &= 2\,\mathrm{sen}\,\theta\cos\theta\end{aligned}}$$

$$\theta = \arcsen\frac{x}{a}$$

DUPL. DEL SENO

$$\cos\theta = \pm\sqrt{1-\mathrm{sen}^2\theta}$$

$$\int \frac{1}{1+\operatorname{sen} x}\,dx =$$

$t = \operatorname{tg}\frac{x}{2}$

$$= \int \frac{1}{1+\frac{2t}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt =$$

$\operatorname{tg}\frac{x}{2} = t$

$\frac{x}{2} = \operatorname{arctg} t$

$\boxed{x = 2\operatorname{arctg} t}$

FORMULE PARAMETRICHE

$\boxed{dx = 2 \cdot \frac{1}{1+t^2}\,dt}$

$$= \int \frac{1}{\frac{1+t^2+2t}{1+t^2}} \cdot \frac{2}{1+t^2}\,dt =$$

$$= \int \frac{\cancel{1+t^2}}{t^2+2t+1} \cdot \frac{2}{\cancel{1+t^2}}\,dt =$$

$\operatorname{sen} x = \frac{2\operatorname{tg}\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}} = \frac{2t}{1+t^2}$

$$= 2\int \frac{1}{\underbrace{t^2+2t+1}}\,dt$$

$$= 2\int \frac{1}{(t+1)^2}\,dt = 2\int (t+1)^{-2}\,dt =$$

$\boxed{\cos x = \frac{1-\operatorname{tg}^2\frac{x}{2}}{1+\operatorname{tg}^2\frac{x}{2}}}$

$$= 2 \cdot \frac{(t+1)^{-1}}{-1} + c =$$

$\int [f(t)]^{\alpha} \cdot f'(t)\,dt$

$= \frac{[f(t)]^{\alpha+1}}{\alpha+1} + c$

$$= -\frac{2}{1+t} + c =$$

$$= \boxed{-\frac{2}{1+\operatorname{tg}\frac{x}{2}} + c}$$

Integrazione per parti

Dividere l'integrale non calcolabile con i soliti metodi in due parti

$y = f(x) \cdot g(x)$ Differenziale del prodotto

$dy = df(x) \cdot g(x) + f(x) \cdot dg(x)$

$-f(x) \cdot dg(x) = df(x) \cdot g(x) - dy$

$f(x)\, dg(x) = dy - df(x) \cdot g(x)$

$\boxed{f(x)\, dg(x) = dy - g(x)\, df(x)}$

Integriamo tutto sia a sinistra che a destra

$\int f(x)\, dg(x) = \int dy - \int g(x)\, df(x)$

$\int f(x)\, dg(x) = y - \int g(x)\, df(x)$

$\boxed{dg(x) = \frac{dg}{dx}\, dx = g'(x)\, dx}$ $\boxed{df(x) = \frac{df}{dx}\, dx = f'(x)\, dx}$

$\boxed{\int f(x) \cdot g'(x)\, dx = f(x) \cdot g(x) - \int g(x)\, f'(x)\, dx}$

ES $\int \log_e x\, dx$ $f(x) = \log_e x$, $g'(x) = 1$

$= f(x) \cdot g(x) - \int g(x)\, f'(x)\, dx =$

$= x \cdot \log_e x - \int \cancel{x} \cdot \frac{1}{\cancel{x}}\, dx$

$= x \log_e x - \int dx = x \log_e - x + c =$

$= x(\log_e x - 1) + c$

$\frac{d\, x(\log_e x - 1)}{dx} = \frac{d\, x \log_e x}{dx} - \cancel{\frac{dx}{dx}}$

$= 1 \cdot \log_e x + \cancel{x} \cdot \frac{1}{\cancel{x}} - 1 =$

$= \log_e x + \cancel{1} - \cancel{1} = \log_e x$

$$\int x \operatorname{sen} x \, dx =$$

$$f(x)g(x) - \int g(x) f'(x) \, dx =$$

$f(x) = x$

$g'(x) = \operatorname{sen} x$

$$= x \cdot (-\cos x) - \int (-\cos x) \cdot 1 \, dx =$$

$$= - x \cos x + \int \cos x \, dx =$$

$$= \underline{-x \cos x + \operatorname{sen} x + C}$$

$$\int x e^x \, dx =$$

$f(x) = x$

$g'(x) = e^x$

$$= f(x) g(x) - \int g(x) f'(x) \, dx =$$

$$= x e^x - \int e^x \cdot dx =$$

$$= x e^x - e^x + C = \underline{e^x (x - 1) + C}$$

$$\int x \cdot \operatorname{arctg} x \, dx =$$

$$\boxed{\begin{array}{l} f(x) = \operatorname{arctg} x \\ g'(x) = x \end{array}}$$

$$= f(x) \cdot g(x) - \int g(x) \cdot f'(x) \, dx =$$

$$\int x \, dx = \frac{x^2}{2}$$

$$= \frac{x^2}{2} \operatorname{arctg} x - \int \frac{x^2}{2} \cdot \frac{1}{1+x^2} \, dx =$$

$$= \frac{x^2}{2} \operatorname{arctg} x - \frac{1}{2} \int \frac{x^2}{1+x^2} \, dx =$$

$$= \frac{x^2}{2} \operatorname{arctg} x - \frac{1}{2} \int \frac{x^2+1-1}{x^2+1} \, dx =$$

$$= \frac{x^2}{2} \operatorname{arctg} x - \frac{1}{2} \int \frac{\cancel{x^2+1}}{\cancel{x^2+1}} \, dx + \frac{1}{2} \int \frac{1}{1+x^2} \, dx =$$

$$= \underline{\frac{x^2}{2} \operatorname{arctg} x} - \frac{1}{2} x + \underline{\frac{1}{2} \operatorname{arctg} x} + C$$

$$\underline{\frac{x^2+1}{2} \operatorname{arctg} x - \frac{1}{2} x + C}$$

$$\int \operatorname{sen}^2 x\,dx = \int \underbrace{\operatorname{sen} x}_{f(x)} \cdot \underbrace{\operatorname{sen} x}_{g'(x)}\,dx =$$

$f(x) = \operatorname{sen} x$

$g'(x) = \operatorname{sen} x$

$$= f(x)\,g(x) - \int g(x)\cdot f'(x)\,dx =$$

$$= \operatorname{sen} x\,(-\cos x) - \int -\cos x \cdot \cos x\,dx =$$

$\boxed{\begin{array}{c} \cos 2x = 2\cos^2 x - 1 \\ \cos^2 x = \dfrac{1+\cos 2x}{2} \end{array}}$

$$= -\operatorname{sen} x \cos x + \int \cos^2 x\,dx$$

$\boxed{\cos^2 x = 1 - \operatorname{sen}^2 x}$

$$= -\operatorname{sen} x \cos x + \int (1 - \operatorname{sen}^2 x)\,dx =$$

$$= -\operatorname{sen} x \cos x + \int dx - \int \operatorname{sen}^2 x\,dx =$$

$$= -\operatorname{sen} x \cos x + x - \int \operatorname{sen}^2 x\,dx$$

$$\int \operatorname{sen}^2 x\,dx = -\operatorname{sen} x \cos x + x - \int \operatorname{sen}^2 x\,dx$$

$$2\int \operatorname{sen}^2 x\,dx = x - \operatorname{sen} x \cos x$$

$$\boxed{\int \operatorname{sen}^2 x\,dx = \frac{x - \operatorname{sen} x \cos x}{2} + C}$$

$$\boxed{f(x) = \ln \frac{x^2+a}{3|x+b|}}$$

$(0;0)$

$\times f(x)$

PASSAGGIO PER $(0;0)$

$$0 = \ln \frac{a}{3|b|}$$

$$1^a \quad \boxed{\ln \frac{a}{3|b|} = 0}$$

$$\lim_{x \to 1} \left(\ln \frac{x^2+a}{3|x+b|} \right) = +\infty$$

$$\ln \left(\frac{1+a}{3|1+b|} \right) \Rightarrow +\infty$$

$$\frac{1+a}{3|1+b|} \to +\infty$$

ASINTOTO VERTICALE

$x = 1$

$$2^a \quad \boxed{3|1+b| = 0}$$

$\ln = \log_e$

$$\begin{cases} \ln \dfrac{a}{3|b|} = 0 \Rightarrow \ln \dfrac{a}{3|b|} = \ln e^0 \Rightarrow \boxed{\dfrac{a}{3|b|} = 1} \\ 3|1+b| = 0 \Rightarrow |1+b| = 0 \Rightarrow 1 + b = 0 \end{cases}$$

$$\boxed{b = -1}$$

$$\frac{a}{3|-1|} = 1 \Rightarrow \frac{a}{3} = 1 \Rightarrow \boxed{a = 3}$$

$$\boxed{\begin{matrix} a = 3 \\ b = -1 \end{matrix}}$$

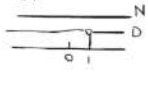
$$\boxed{y = \ln \frac{x^2+3}{3|x-1|}}$$

$$\text{Dom} f \begin{cases} \dfrac{x^2+3}{3|x-1|} > 0 \quad ① \qquad \forall x \in \mathbb{R} \setminus \{1\} \\ 3|x-1| \neq 0 \Rightarrow x \neq 1 \end{cases}$$

$$\frac{x^2+3}{3|x-1|} > 0$$

N $x^2 + 3 > 0$

$x^2 > -3 \quad \forall x \in \mathbb{R}$



D $|x-1| > 0 \qquad 0 > 0 ?!!$

$\forall x \in \mathbb{R} \setminus \{1\}$

$\text{Dom} f = \mathbb{R} \setminus \{1\}$